Security of Shamir's Secret-sharing against Physical Bit Leakage: Secure Evaluation Places

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Abstract. NIST aims to recommend best practices to make secret-sharing schemes like Shamir's and the additive secret-sharing schemes more secure. Side-channel attacks that leak one physical bit from every secret share completely break the additive secret-sharing scheme – leaking their least significant bit suffices. Shamir's secret-sharing scheme inherits these vulnerabilities if its evaluation places are carelessly chosen. To further NIST's efforts in this context, it is natural to investigate (a) which evaluation places of Shamir's secret-sharing scheme are robust and (b) which evaluation places are vulnerable to such attacks.

A random choice of evaluation places is robust to such leakages with high probability. However, adversarially chosen randomness defeats such randomized constructions because techniques to distinguish secure evaluation places from insecure ones are unknown.

Our work introduces the following technical innovations.

1. A modulus choice for which protection against the LSB attack ensures protection against any physical bit attack.
2. An algorithm to efficiently identify secure and vulnerable evaluation places against the LSB attack.

Building on these, we recommend modulus and evaluation places that make Shamir's secret-sharing scheme robust to physical bit leakage – the first complete derandomization of existing randomized constructions.

Our work introduces new techniques to analyze the security of secret-sharing schemes. It connects the security of secret-sharing schemes to the orthogonality/independence properties of a system of square wave functions. The quality of this connection depends on finding good simultaneous rational approximations – a Dirichlet-type approximation problem efficiently solved using the LLL algorithm.

Keywords: Shamir secret-sharing, physical bit leakage, secure evaluation places, local leakage resilience, derandomization, square wave families, Fourier analysis

1 Introduction

The National Institute of Standards and Technology (NIST) recently called for submissions for future recommendations and guidelines to make threshold

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schemes more secure [10]. The additive and Shamir's secret-sharing schemes are fundamental building blocks with numerous applications in threshold cryptography and distributed computing [4], like secure storage and computation over secrets and building sophisticated public-key primitives. The insecurity of these fundamental building blocks endangers any primitive built on top of them.

Probing wires and introducing random faults into them seem innocuous but lead to devastating attacks – the more straightforward the attack, the greater a security threat it poses. For example, Boneh et al. [9] showed the vulnerability of computing RSA signatures to random fault injection into memory. Ishai, Sahai, and Wagner [24] introduced the bit probing model to theoretically investigate threats posed by an adversary that can probe a bounded number of memory locations. The additive and Shamir's secret-sharing schemes have been used to design masking schemes as countermeasures to side-channel attacks in this model [24, 42, 17]. This work studies the security of secret-sharing schemes when an adversary can probe physical bits from memory that stores the secret shares in their binary representation.

The standard notion of security for secret-sharing schemes considers an adversary who obtains some secret shares and has no information about the remaining secret shares. Innovative side-channel attacks have repeatedly violated this "all or nothing" corruption assumption (starting with the works of Kocher et al. [26, 27, 12]). We consider an adversary that can probe a few physical bits from all secret shares in the physical bit probe model. For example, the adversary can leak every secret share's least significant bit (LSB). A secret-sharing scheme is robust to such probing attacks if the leakage remains statistically independent of the secret [5, 18].

The LSB attack breaks the additive secret-sharing scheme's security [33, 35]. For any finite field, there are two secrets that the adversary can distinguish with a constant $(2/\pi)^n \approx (0.63)^n$ advantage. Shamir's secret-sharing scheme presents a (potentially) secure alternative to the additive secret-sharing scheme. However, it inherits the additive secret-sharing scheme's vulnerabilities if the evaluation places are carelessly chosen [33, 14]. A random choice of evaluation places is secure with high probability against physical bit probing attacks [33]. However, such randomized constructions are vulnerable to adversarially chosen randomness because algorithms to distinguish secure evaluation places from insecure ones are unknown. Surprisingly, even against the LSB leakage attack, such a classification algorithm is unknown, let alone against general physical bit leakage. This is the classical challenge of searching hay in the haystack (or finding water in the ocean) problems [46].

Model. Our work studies Shamir's secret-sharing scheme over prime fields among n parties with reconstruction threshold k. We study the security of such secret-sharing schemes against an adversary who leaks one physical bit from every secret share. The security parameter λ represents the number of bits in the binary representation of a secret share.

Our results. Our work identifies (a) secure evaluation places for Shamir's secret-sharing scheme and (b) new leakage threats. We perform concrete security analysis – not just asymptotic analysis.

Based on our technical contributions, we recommend choosing prime fields of order $p = 2^\lambda - 1$, i.e., Mersenne primes. We present efficient algorithms to identify secure evaluation places for Shamir's secret-sharing scheme for (1) $(n,k) = (2,2)$, (2) $(n,k) = (3,2)$, and (3) the general $n = k$ cases. To begin, we present efficient algorithms to identify secure evaluation places for the $(n,k) = (2,2)$ and $(n,k) = (3,2)$ cases. The statistical distance between the leakage distributions for two secrets is at most $1/\sqrt{p}$ for these secure evaluation places. Our algorithms identify at least $1 - 1/\sqrt{p}$ fraction of all secure evaluation places. Furthermore, we also present explicit evaluation places secure for any Mersenne prime p. Finally, we lift the security of the $(n,k) = (2,2)$ base scheme to the security of Shamir's secret-sharing involving more parties, i.e., the general $n = k > 2$ case.

Technical innovations. An "exponential-type" sum involving an oscillatory function over the finite field F captures the statistical distance between the leakage for two different secrets. Accurately estimating this sum is challenging. We establish a new connection to a family of *periodic square waves* to estimate this sum. Families of square waves like the ones by Haar [19], Walsh [48], and Rademacher [41] are central to engineering and science. In our work, we connect the leakage resilience of secret-sharing schemes with the properties of another family of square waves (see, for example, [47, 22, 21]) The similarity between the square waves and their offsets, captured by an integral, serves as a proxy to estimate this sum. Choosing an appropriate basis (by solving a Dirichlet-type approximation problem using the LLL algorithm [31]) improves the accuracy of this estimation strategy. In the context of security analysis of secret-sharing schemes, these analysis techniques are new and possibly of broader interest.

Remark 1 (Shortcomings of Randomized Constructions and Necessity of Complete Derandomization). We completely derandomize the randomized construction of [33] when (a) $(n,k) = (2,2)$, (b) $(n,k) = (3,2)$, and (c) $n = k > 2$. [33] proved that a random set of (distinct) evaluation places is secure, with high probability. However, parties cannot determine whether the selected evaluation places are secure or not. Consequently, even though nearly all evaluation places are secure, adversarially instantiated randomness could lead to choosing insecure evaluation places unbeknownst to the parties. In mathematics and computer science, it is relatively common to design randomized experiments where most objects in the universe possess a "desired property." However, it is computationally infeasible to determine whether a given object has the desired property. For example, in designing good expanders, randomness extractors, and linear codes. In the context of the leakage resilience of Shamir's secret-sharing scheme, our efficient algorithms identify nearly all secure evaluation places. Additionally, we also present explicit secure evaluation places.

Remark 2 (Constructing new Leakage-resilient Secret-sharing Schemes: Concerns). A highly influential sequence of works has constructed new secret-sharing schemes resilient to leakage attacks [7, 2, 45, 3, 29, 8, 15, 16, 23, 13, 38, 11]. However, these new constructions have worse information rates and lack desirable algebraic properties (like linearity and multiplication-friendliness). Furthermore, replacing all deployed instances of the additive and Shamir's secret-sharing scheme from applications seems insurmountable.

1.1 Basic Terminology and Definition

We introduce a few definitions to facilitate the presentation of our results.

Shamir's secret-sharing scheme. Shamir's secret-sharing scheme among n parties with reconstruction threshold k over a finite field F proceeds as follows. Fix a secret $s \in F$. Consider arbitrary distinct evaluation places $\alpha_1, \alpha_2, \dots, \alpha_n \in F^*$. Sample a random F-polynomial $P(X)$ such that $\deg P < k$ and $P(0) = s$. The secret shares are $s_1 = P(\alpha_1)$, $s_2 = P(\alpha_2)$, $\dots$, and $s_n = P(\alpha_n)$. This secret-sharing scheme is denoted by $\mathsf{ShamirSS}(n, k, \vec{\alpha})$. The joint distribution of the secret shares is $\mathsf{Share}(s)$ – other parameters will be clear from the context.

Representation. The secret shares are stored in their natural binary representation. For the prime field F_p (of order p), the elements of this field correspond to the binary representation of the elements $\{0, 1, \dots, (p-1)\}$. The security parameter λ is the number of bits in the binary representation. For example, in the case of a prime field F_p, we have $2^{\lambda-1} < \mathsf{card}(F_p) = p < 2^{\lambda}$.

Leakage Functions. This work considers physical bit leakage. So, $\mathrm{LSB}_i \colon F \to \{0, 1\}$ is the function that outputs the i-th least significant bit. For example, LSB_0 (or, LSB, for brevity) outputs 0 for the elements $\{0, 2, \dots, (p-1)\}$, where F is a prime field of order $p \geqslant 3$. Similarly, LSB_1 outputs 0 for the elements $\{0, 1, 4, 5, 8, 9, \dots\}$. $\vec{\mathrm{LSB}}_{i_1,\dots,i_n} := (\mathrm{LSB}_{i_1}, \mathrm{LSB}_{i_2}, \dots, \mathrm{LSB}_{i_n})$ represents a leakage function, where $i_1, \dots, i_n \in \{0, 1, \dots, \lambda-1\}$. For $k \in \{1, 2, \dots, n\}$, this leakage function will leak the i_k-th LSB from the k-th secret share. The joint distribution of the leakage is $\vec{\mathrm{LSB}}_{i_1,\dots,i_n}(\mathsf{Share}(s))$.

Leakage Resilience. The insecurity of $\mathsf{ShamirSS}(n, k, \vec{\alpha})$ against a family of leakage attacks $\mathcal{F}$ is

$$\varepsilon_{\mathcal{F}}(\vec{\alpha}) := \max_{s \in F^*, f \in \mathcal{F}} \mathsf{SD}\left(f(\mathsf{Share}(0)) \ , \ f(\mathsf{Share}(s)) \right).$$

This work considers two families of leakage functions: (1) PHYS: The family of all physical bit leakage functions and (2) LSB: The LSB leakage attack.

1.2 Our Results

This section presents our results for the (1) $(n, k) = (2, 2)$, (2) $(n, k) = (3, 2)$, and (3) $n = k > 2$ cases. Our objective is to efficiently identify secure evaluation places (i.e., ones with low insecurity) and demonstrate efficient attacks on vulnerable evaluation places (i.e., ones with high insecurity). Below, for $x, y, z \in \mathbb{R}$, the expression $x = y \pm z$ is a succinct representation for $x \in [y - z, y + z]$.

Result 1: Security against Physical Bit Leakage when $(n,k) = (2,2)$. For the $n = k = 2$ case, we recommend using field F_p, where p is a Mersenne prime – a prime of the form $2^\lambda - 1$. In this context, our work proves the following results.

1. Corollary 4: Given evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2)$ as input, we efficiently compute a closed-form estimate $\varepsilon^{(\mathrm{OUR})}_{\mathrm{PHYS}}(\vec{\alpha}) \in [0,1]$ satisfying
$$\varepsilon^{(\mathrm{OUR})}_{\mathrm{PHYS}}(\vec{\alpha}) = \varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \quad \pm \left(\frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}\right).$$
Intuitively, our estimation $\varepsilon^{(\mathrm{OUR})}_{\mathrm{PHYS}}(\vec{\alpha})$ is within an additive error of $\mathcal{O}\big(1/\sqrt{p}\big)$ of the actual insecurity $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha})$. The results below analyze this accurate estimation as a proxy for actual insecurity.
2. Corollary 5: Using our estimation, Figure 1 presents our efficient algorithm to classify $\vec{\alpha}$ as secure or not. If our algorithm classifies $\vec{\alpha}$ as secure, then
$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{1+8^{5/4}}{\sqrt{p}} \quad + \frac{13/2}{p},$$
which is exponentially small in the security parameter λ. Among all possible distinct evaluation places, our algorithm classifies (at least)
$$1 \quad - \frac{1}{4\ln 2} \cdot \frac{(\ln p)^2}{\sqrt{p}} \Big(1 + \mathrm{o}(1)\Big)$$
fraction of them as secure. Supporting Material G enumerates all secure evaluation places for a small Mersenne prime $p = 2^{13} - 1$ using our algorithm.
3. Corollary 6: Using our estimation technique, we present an efficient adversary that generates $(s, f) \in F^* \times \mathcal{F}$ such that it distinguishes the secret 0 from secret s by leaking f from the secret shares with an advantage
$$\geqslant \varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \quad - \frac{2 \cdot 8^{5/4}}{\sqrt{p}} - \frac{13}{p}.$$
Intuitively, for vulnerable evaluation places $\vec{\alpha}$, we present an efficient leakage attack that achieves a comparable distinguishing advantage. For example, evaluation places $\vec{\alpha} = (1,2)$ and $\vec{\alpha} = (1,3)$ have insecurity (roughly) 1 and $1/3$, respectively.
4. Corollary 7: We present explicit evaluation places that are secure against physical bit leakages. If evaluation places (α_1, α_2) satisfy $\alpha_2 \cdot \alpha_1^{-1} = 2^{\lfloor \lambda/2 \rfloor} - 1$, then
$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{4 \cdot \left(2^{\lfloor \lambda/2 \rfloor} + 2^{\lceil \lambda/2 \rceil}\right) - 6}{p} = \mathcal{O}\left(\frac{1}{\sqrt{p}}\right).$$
The upper bound is meaningful (i.e., less than 1) for all $\lambda \geqslant 7$ (Mersenne prime $p \geqslant 127$).

Section 5 presents the corollary statements and their proofs (Section 5.3 to Section 5.6 state and prove Corollary 4 to Corollary 7, respectively).

Technical Result: Security against LSB Leakage when $(n,k)=(2,2)$. The above-mentioned results on physical bit leakage bootstrap from similar results against the LSB leakage. The results in this section hold for any prime field F_p, where $p \geqslant 3$, not just for Mersenne primes.

1. Corollary 1: Given evaluation places $\vec{\alpha}=(\alpha_1,\alpha_2)$ as input, we efficiently compute a closed-form estimate $\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha}) \in [0,1]$ satisfying
$$\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha}) = \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \quad \pm \left(\frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}\right).$$
2. Corollary 2: Using the estimation above, Figure 2 presents an efficient algorithm to classify $\vec{\alpha}$ as secure or not. If our algorithm classifies $\vec{\alpha}$ as secure, then
$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \leqslant \frac{1+8^{5/4}}{\sqrt{p}} \quad + \frac{13/2}{p},$$
which is exponentially small in the security parameter λ. Among all possible distinct evaluation places, our algorithm classifies (at least)
$$1 \quad - \frac{\ln p}{4\sqrt{p}} - \frac{5/2}{\sqrt{p}}$$
fraction of them as secure.
3. Corollary 3: Using our estimation technique, we present an efficient adversary that generates $s \in F^*$ such that it distinguishes the secret 0 from secret s by leaking the LSB of each secret share with an advantage
$$\geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \quad - \frac{2 \cdot 8^{5/4}}{\sqrt{p}} - \frac{13}{p}$$
Therefore, if the insecurity $\varepsilon_{\mathrm{LSB}}(\vec{\alpha})$ is large, then our efficient leakage attack achieves a comparable distinguishing advantage.
For example, evaluation places $\vec{\alpha}=(1,2)$ and $\vec{\alpha}=(1,3)$ have insecurity (roughly) 0 and $\geqslant \cos^2(\pi/2p)\cdot(1/3)$, respectively. Note that $\vec{\alpha}=(1,2)$ is secure against LSB leakage but not against physical bit leakage. However, $\vec{\alpha}=(1,3)$ is insecure against the LSB attack and, hence, against general physical bit leakage.

Section 4 presents the corollary statement and their proofs (Section 4.1 to Section 4.3 state and prove Corollary 1 to Corollary 3, respectively).

Result 2: Security against Physical Bit Leakage when $n=k>2$. We recommend using a prime field F_p, such that $p=2^\lambda-1$. Corollary 8 presents an efficient (randomized) algorithm to choose evaluation places $\vec{\alpha}$ such that the corresponding $\mathsf{ShamirSS}(n,n,\vec{\alpha})$ is secure to physical bit leakages; the insecurity is at most $1/\sqrt{p}$. One can identify when this algorithm fails to choose secure $\vec{\alpha}$, and this failure probability is exponentially small in the security parameter λ. Using repeated sampling, the failure probability can be further reduced exponentially. Section 6 presents Theorem 2, which implies this corollary. Supporting Material D proves this theorem using Fourier analysis. Section 6.1 presents the proof of Corollary 8.

Result 3: Security against Physical Bit Leakage when $(n, k) = (3, 2)$. Corollary 8 presents a technique to "lift" the security of two evaluation places to the security of n evaluation places $(\alpha_1, \alpha_2, \dots, \alpha_n)$. It maps $\vec{\alpha}$ to evaluation places $\vec{\beta} = (\beta_1, \dots, \beta_n)$ such that if $\mathsf{ShamirSS}(2, 2, \ (\beta_1, \beta_2)\ )$ has ε insecurity against physical bit leakage attacks then the $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ has (at most) 2ε insecurity against physical bit leakage attacks. This mapping $\vec{\alpha} \mapsto \vec{\beta}$ is highly nontrivial and depends on the properties of generalized Reed Solomon codes. A natural question arises: Are there more natural lifting techniques?

For example, consider the following natural lifting technique: If all evaluation pairs (α_i, α_j) are secure then is $\vec{\alpha}$ also secure? It is unclear whether evaluation places (α_i, α_j) would retain their security in the presence of additional leakage from other secret shares. We prove this result for $(n, k) = (3, 2)$.

Consider distinct evaluation places $(\alpha_1, \alpha_2, \alpha_3)$ as follows. Suppose the $\mathsf{ShamirSS}(2, 2, \ (\alpha_i, \alpha_j)\ )$ secret sharing scheme has ε insecurity against physical bit leakages, for all distinct $i, j \in \{1, 2, 3\}$. Lemma 9 proves that the $\mathsf{ShamirSS}(3, 2, \ (\alpha_1, \alpha_2, \alpha_3)\ )$ has (at most) 3ε insecurity against physical bit leakages.

The key technical contribution of this specialized lifting theorem is the following observation. The statistical distance between two leakage distributions has a "three-wise correlation term." We prove that this correlation term is independent of the secret, even though $k = 2$ is less than the degree of the correlation, which is 3.

For the converse, note that if there are insecure evaluation places (α_i, α_j), then the entire $\mathsf{ShamirSS}(3, 2, \vec{\alpha})$ is also vulnerable.

Table 2 in Supporting Material G presents secure evaluation places $(\alpha_1, \alpha_2, \alpha_3)$ when $\alpha_1 = 1$, $\alpha_2 = 95$. The exhaustive list is too long to include in the paper. Finding general evaluation places that are secure for $(n, k) = (3, 2)$ (in the spirit of Corollary 7) is an open problem.

1.3 Organization of the Paper

1. Section 2 presents a high-level overview of our technical approach to the derandomization problem.
2. Section 3 presents the preliminaries.
3. Section 4 states and proves our results pertaining to LSB leakage.
4. Section 5 states and proves our results for general physical bit leakages.
5. Section 6 lifts the $(n, k) = (2, 2)$ results to more general $n = k > 2$.
6. Section 7 presents our results for $(n, k) = (3, 2)$ case.
7. Section 8 summarizes the prior relevant works.
8. Section 9 mentions open problems and technical challenges in characterizing the security of leakage-resilient secret-sharing schemes.

2 Technical Overview

This section outlines our technical approach for some representative results.

2.1 Result 1: Physical Bit Leakage $(n, k) = (2, 2)$

Suppose the evaluation places are $\vec{\alpha} = (\alpha_1, \alpha_2)$. Our objective is to *determine whether Shamir's secret-sharing scheme with these evaluation places is secure against all physical bit leakage attacks.*

For $i, j \in \{0, 1, \dots, \lambda - 1\}$, consider the leakage attack $\vec{\mathrm{LSB}}_{i,j}$. This leakage attack leaks the i-th LSB of the secret share 1 and the j-th LSB of the secret share 2. We prove that, for a Mersenne prime $p = 2^\lambda - 1$, the security of $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ against the $\vec{\mathrm{LSB}}_{i,j}$ leakage is equivalent to the security of $\mathsf{ShamirSS}(2, 2, \vec{\alpha'})$ against the LSB attack, where $\alpha'_1 = 2^{-i}\alpha_1$ and $\alpha'_2 = 2^{-j}\alpha_2$ (see Lemma 7). Consequently, it suffices to test the security of the evaluation places $(2^{-i}\alpha_1, 2^{-j}\alpha_2)$ against the LSB attacks, for all $i, j \in \{0, 1, \dots, \lambda - 1\}$.

For each i, j, the call to the "LSB security check subroutine" identifies evaluation places potentially vulnerable to the $\vec{\mathrm{LSB}}_{i,j}$ leakage attack. Using a naïve union bound, the total number of potentially vulnerable evaluation places would be proportional to λ^2. However, using properties of Shamir's secret-sharing scheme, one can improve upon this naïve estimate, which is a significant overestimation of the actual number of vulnerable evaluation places.

We use a "normalization result" to improve this bound. Properties of the generalized Reed Solomon codes imply that $\mathsf{ShamirSS}(n, k, \vec{\gamma})$ is identical to $\mathsf{ShamirSS}(n, k, \Lambda \cdot \vec{\gamma})$, for any $\Lambda \in F^*$, evaluation places $\vec{\gamma}$, and $n, k \in \{1, 2, \dots\}$ (see Lemma 10 in Supporting Material B). Therefore, it suffices to test the security of the evaluation places $(2^k\alpha_1, \alpha_2)$ against the LSB attack, for all $k \in \{0, 1, \dots, \lambda - 1\}$. As a result, only a linear number of calls are made to the LSB security testing algorithm instead of the naïve quadratic calls.

Figure 1 presents this pseudocode. The next section presents the pseudocode to determine security against the LSB attack.

Remark 3 (An Edge Case). The algorithm determining the security of Shamir's secret-sharing scheme to LSB attack requires the evaluation places to be distinct. Even though α_1 and α_2 are distinct, it may be the case that $2^k\alpha_1 = \alpha_2$, for some $k \in \{0, 1, \dots, \lambda - 1\}$. So, the call to the "LSB security check subroutine" with argument $(2^k\alpha_1, \alpha_2)$ would be invalid. Lemma 6 proves that this edge case is insecure. This case captures why evaluation places $(1, 2)$ are insecure against physical bit leakage.

2.2 Technical Result: LSB Leakage $(n, k) = (2, 2)$.

Suppose the evaluation places are $\vec{\alpha} = (\alpha_1, \alpha_2)$. Our objective is to *determine whether these evaluation places are secure against the* LSB *attack*. The presentation below is for all prime fields F of order p such that $p \geqslant 3$ – not just for a Mersenne prime p.

Input. Distinct evaluation places $\alpha_1, \alpha_2 \in F^*$, and F is the finite field of order p, a Mersenne prime

Output. Decide whether the evaluation places (α_1, α_2) are secure to all physical bit leakage attacks

Algorithm.

1. If there is $k \in \{0, 1, \ldots, \lambda - 1\}$ such that $2^k \alpha_1 = \alpha_2$: Return insecure
2. For $k \in \{0, 1, \ldots, \lambda - 1\}$:
   (a) Call the algorithm in Figure 2 with evaluation places $(2^k \alpha_1, \alpha_2)$
   (b) If the algorithm returns "may be insecure," return may be insecure
3. Declare $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ is secure against physical bit attacks.

Fig. 1. Identify secure evaluation places for Shamir's secret-sharing scheme against all physical bit leakage attacks.

Consider a secret $s \in F^*$. Our objective is to estimate the statistical distance between the joint leakage distributions when (a) the secret is 0 and (b) the secret is s. Using a combinatorial argument, we prove that the statistical distance is identical to the following expression (see Lemma 2) for an appropriate Δ.

$$\frac{1}{2p} \cdot \left| \Sigma^{(0)}_{\alpha_1, \alpha_2} - \Sigma^{(\Delta)}_{\alpha_1, \alpha_2} \right|,$$

where

$$\Sigma^{(\Delta)}_{k,\ell} := \sum_{T \in F} \mathrm{sign}_p(kX) \cdot \mathrm{sign}_p(\ell(X - \Delta))$$

$$\mathrm{sign}_p(X) := \begin{cases} +1, & \text{if } X \in \{0, 1, \ldots, (p-1)/2\} \mod p \\ -1, & \text{if } X \in \{-(p-1)/2, \ldots, -1\} \mod p. \end{cases}$$

The $s \mapsto \Delta$ mapping is a linear automorphism over F.

Remark 4 (Intuition of the Expression). The elements $\{0, 1, \ldots, (p-1)/2\} \subseteq F_p$ are positive elements, and the remaining elements are negative. We are considering functions that are the "signs of lines." For example, $\mathrm{sign}_p(kX)$ is the sign of the line $Y = kX$ over the finite field, which is an oscillating function. Likewise, $\mathrm{sign}_p(\ell(X - \Delta))$ is the sign of the (affine) line $Y = \ell(X - \Delta)$ over the finite field, another oscillating function. The expression $\Sigma^{(\Delta)}_{k,\ell}$ – the inner product of these two functions – measures the correlation between these two functions. Leakage resilience to LSB attacks is equivalent to this correlation being independent of the secret s and, in turn, the parameter Δ.

The evaluation places $\vec{\alpha}$ are secure if (and only if) $\frac{1}{2p} \cdot \left| \Sigma^{(0)}_{\alpha_1, \alpha_2} - \Sigma^{(\Delta)}_{\alpha_1, \alpha_2} \right|$ is small for all $\Delta \in F^*$. To this end, we aim to estimate $\frac{1}{p} \cdot \Sigma^{(\Delta)}_{\alpha_1, \alpha_2}$, for all $\Delta \in F$. This expression is the sum of an oscillating function that appears challenging to estimate accurately.

Our technical solution's innovation is to estimate (1) a real extension of this function using integration and (2) establish a connection between the sum of the oscillating function and the integration. Toward the first sub-objective, the integral is defined as follows.

$$\begin{aligned} I_{k,\ell}^{(\delta)} &:= \int_0^1 \varphi(kt)\cdot\varphi(\ell(t-\delta))\,\mathrm{d}t \\ \varphi(x) &:= \operatorname{sign}\sin(2\pi x) \\ \operatorname{sign}(x) &:= \begin{cases} +1, & \text{if } x>0 \\ 0, & \text{if } x=0 \\ -1, & \text{if } x<0. \end{cases} \end{aligned}$$

The function $\varphi(kx)$ is a *square wave function* with period 1 for all $k \in \mathbb{Z}$. This function is the real extension of the "sign of lines" function $\operatorname{sign}_p(X)$ above (with a scaling factor). The connection is that $\operatorname{sign}_p(X) = \varphi(x)$, where $x = X/p$ and $X \in F^*$. The square wave family $\{\varphi(kx)\}_{k\in\{1,2,\dots\}}$ have been studied in the literature [47, 22, 21]. However, only $I_{k,\ell}^{(0)}$ was determined. Our work presents a closed-form expression for $I_{k,\ell}^{(\delta)}$, for all $\delta \in [0,1]$ (see Lemma 5).

Now, the second sub-objective is to estimate $\frac{1}{p}\cdot\Sigma_{k,\ell}^{(\Delta)}$ using the integral $I_{k,\ell}^{(\delta)}$, where $\delta := \Delta/p$. The accuracy of estimating the sum of an oscillating function using its integral depends on how many times the function oscillates. The number of oscillations of the function $\operatorname{sign}_p(kX)\cdot\operatorname{sign}_p(\ell(X-\Delta))$ is proportional to $\left(|k|_p + |\ell|_p\right)/p$, where the "norm- mod p" function is defined below.

$$|X|_p := \begin{cases} X', & \text{if } X = X' \mod p \text{ and } X' \in \{0,1,\dots,(p-1)/2\}, \\ -X', & \text{if } X = X' \mod p \text{ and } X' \in \{-(p-1)/2,\dots,-1\}. \end{cases}$$

The estimation error is $(|k|_p + |\ell|_p)/p$ and will drown the value of the integral for large $|k|_p + |\ell|_p$.

At this point, the "normalization result" from the previous section is useful. The security of $\mathsf{ShamirSS}(2,2,\vec{\alpha})$ is identical to the security of $\mathsf{ShamirSS}(2,2,\vec{\gamma})$, if $\vec{\gamma} = \Lambda\cdot\vec{\alpha}$ and $\Lambda \in F^*$. So, instead of estimating $\frac{1}{p}\cdot\Sigma_{\alpha_1,\alpha_2}^{(\Delta)}$, we estimate $\frac{1}{p}\cdot\Sigma_{u,v}^{(\Delta)}$, where $vu^{-1} = \alpha_2\alpha_1^{-1}$ and $|u|_p, |v|_p$ are small. Dirichlet's approximation theorem [43, 44] ensures that there are u and v such that $|u|_p, |v|_p$ are at most $\sqrt{p}$. However, finding this (u,v) is computationally inefficient. We efficiently solve this problem with (a small) constant multiplicative slack using the LLL algorithm [31].

To conclude, given (α_1,α_2), we use the LLL algorithm to construct appropriate "small norm" (u,v). Next, we use the closed-form expressions for $I_{u,v}^{(0)}$ and $I_{u,v}^{(\delta)}$ to estimate $\frac{1}{2p}\left|\Sigma_{\alpha_1,\alpha_2}^{(0)} - \Sigma_{\alpha_1,\alpha_2}^{(\Delta)}\right|$, where $\delta = \Delta/p$. Finally, we maximize over $\Delta \in F^*$ and determine the insecurity of the evaluation places (α_1,α_2) against the LSB attack. We also present the closed-form expressions for the maximum value, which our decision algorithm directly uses. Figure 2 presents the pseudocode of this algorithm.

Input. Distinct evaluation places $\alpha_1, \alpha_2 \in F^*$

Output. Decide whether the evaluation places (α_1, α_2) are secure to the LSB leakage attack

Algorithm.

1. Define the equivalence class
$$[\alpha_1 : \alpha_2] := \Big\{ (u, v) \colon u = \Lambda \cdot \alpha_1, v = \Lambda \cdot \alpha_2, \Lambda \in F^* \Big\}.$$
Use the LLL [30] algorithm to (efficiently) find $(u, v) \in [\alpha_1 : \alpha_2]$ such that
$$u, v \in \{-B, -(B-1), \ldots, 0, 1, \ldots, (B-1), B\} \mod p,$$
where $B := \left\lceil 2^{3/4} \cdot \sqrt{p} \right\rceil$. For completeness, Figure 5 in Supporting Material A presents this algorithm.
2. Remark: Henceforth, our algorithm interprets $u, v \in \{-B, \ldots, 0, 1, \ldots, B\} \mod p$ as integers.
3. Compute $g = \gcd(u, v)$.
4. If $u \cdot v/g^2$ is even: Declare $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ is <u>secure</u> to LSB leakage attacks
5. (Else) If $u \cdot v/g^2$ is odd and $\left|u \cdot v/g^2\right| \geqslant \sqrt{p}$: Declare $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ is <u>secure</u> to LSB leakage attacks
6. (Else) Declare that the security of $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ against LSB attacks <u>may be insecure</u>

Fig. 2. Identify secure evaluation places for Shamir's secret-sharing scheme against the LSB leakage attack.

The insecurity is at most $(|u|_p + |v|_p)/p = \mathcal{O}(1/\sqrt{p})$ for secure evaluation places, which is exponentially small in the security parameter. Our analysis identifies the concrete constants. Our analysis is tight and, consequently, also identifies new leakage attacks for insecure evaluation places.

Remark 5 (Minor Subtlety). Observe that $\mathrm{sign}_p(0) = +1$ but $\mathrm{sign}(0) = 0$. Using careful accounting, we show that the impact of this disagreement is only $\pm 1/p$ in the overall insecurity estimation.

2.3 Result 2: Physical Bit Leakage $n = k > 2$

Our objective is to choose n distinct evaluation places $\alpha_1, \alpha_2, \ldots, \alpha_n \in F^*$ such that the corresponding $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ is secure against physical bit leakage attacks. We prove a lifting theorem (Theorem 2) that proves the following result.

Given, evaluation places $\vec{\alpha}$ we define new evaluation places

$$\beta_i := \left(\alpha_i \prod_{j \neq i} (\alpha_i - \alpha_j)\right)^{-1}.$$

Now consider the $\mathsf{ShamirSS}(2, 2, (\beta_i, \beta_j))$ secret-sharing scheme for all distinct $i, j \in \{1, 2, \ldots, n\}$. If one of these secret-sharing schemes is secure against physical bit leakage, then the $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ secret-sharing scheme is also secure. More concretely, if the insecurity of $\mathsf{ShamirSS}(2, 2, (\beta_i, \beta_j))$ is (at most) ε, for some distinct $i, j \in \{1, 2, \ldots, n\}$, then the $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ secret-sharing scheme is (at most) 2ε insecure.

We already have an efficient algorithm to classify evaluation places of $\mathsf{ShamirSS}(2, 2, (\beta_i, \beta_j))$ as secure or not. We can use this algorithm to detect whether our chosen $\vec{\alpha}$ has such a secure (β_i, β_j) pair of evaluation places. The proof of this result is entirely Fourier-analytic, and it is presented Supporting Material D.

2.4 Result 3: Physical Bit Leakage $(n, k) = (3, 2)$

Suppose the evaluation places are $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. Our objective is to determine whether these evaluation places are secure against all physical bit leakage attacks.

A necessary condition is that $\mathsf{ShamirSS}(2, 2, (\alpha_i, \alpha_j))$ must be secure, for distinct $i, j \in \{1, 2, 3\}$. Surprisingly, we prove that this condition is essentially sufficient. Technically, we shall prove that the "three-wise correlation" among the three leakage bits is statistically independent of the secret.

Using the triangle inequality, we prove that the statistical distance between the joint leakage distributions for (a) the secret 0 and (b) secret $s \in F^*$ is upper-bounded by the sum of four terms.

1. Three terms corresponding to
$$\frac{1}{2p}\left|\Sigma^{(0)}_{\alpha_i,\alpha_j} - \Sigma^{(\Delta_{i,j})}_{\alpha_i,\alpha_j}\right|,$$
for distinct $i, j \in \{1, 2, 3\}$ and appropriate $\Delta_{i,j}$ determined by a linear automorphism $s \mapsto \Delta_{i,j}$. These terms ensure that the leakage from any two secret shares is statistically independent of the secret.
2. The final term corresponds to
$$\frac{1}{2p}\left|\Sigma^{(0,0)}_{\alpha_1,\alpha_2,\alpha_3} - \Sigma^{(\Delta,\Delta')}_{\alpha_1,\alpha_2,\alpha_3}\right|,$$
where
$$\Sigma^{(\Delta,\Delta')}_{k,\ell,m} := \sum_{T \in F} \mathrm{sign}_p(kT) \cdot \mathrm{sign}_p(\ell(T - \Delta)) \cdot \mathrm{sign}_p(m(T - \Delta'))$$
and the mappings $s \mapsto \Delta$ and $s \mapsto \Delta'$ are two linear automorphisms. This term captures the three-wise correlation between the leakage bits. We prove that the contribution of this term is $\pm\frac{1}{p}$, which is exponentially small in the security parameter.

What did we gain by proving that the three-wise correlation between the leakage bits is statistically independent of the secret? Without this independence property, we would be forced to estimate this expression using an integral. The error in this estimation would be proportional to $(|u|_p + |v|_p + |w|_p)/p$, where $(u, v, w) \in [\alpha_1 \colon \alpha_2 \colon \alpha_3]$. Using Dirichlet's approximation theorem [31], one can only ensure that $|u|_p, |v|_p, |w|_p \leqslant p^{2/3}$ simutaneously. Consequently, our estimation error will be of the order $p^{-1/3}$. Therefore, we would only be able to guarantee that insecurity is $\leqslant p^{-1/3}$. Note that currently, we are able to ensure that the insecurity is $\leqslant p^{-1/2} \ll p^{-1/3}$.

Remark 6 (Odd-wise Correlation). In general, the $(2t+1)$-wise correlation terms contribute at most $\pm \frac{t}{p}$ to the statistical distance, where $t \in \{0, 1, \dots\}$.

3 Preliminaries

For real numbers $a \leqslant b$ and ε, the notation $[a, b] \pm \varepsilon$ represents the interval $[a-\varepsilon, b+\varepsilon]$. For brevity, $a \pm \varepsilon$ represents $[a, a] \pm \varepsilon$, which is the interval $[a-\varepsilon, a+\varepsilon]$. For ease of readability, we write $x = a \ \pm \varepsilon$ to indicate $x \in a \pm \varepsilon$.

For a set S, $\mathsf{card}(S)$ represents its cardinality. For $S \subseteq F$ and $x \in F$, we denote $S \cdot x$ as the set $\{s \cdot x : s \in S\}$. For $S \subseteq F$, the function $\mathbb{1}_S \colon F \to \{0, 1\}$ is the indicator function of the set S: $\mathbb{1}_S(x) = 1$, if $x \in S$; otherwise, $\mathbb{1}_S(x) = 0$.

For functions $f, g \colon \mathbb{N} \to \mathbb{N}$, we say $f(\lambda) \sim g(\lambda)$ if $f(\lambda) = g(\lambda) \cdot (1 + \mathrm{o}(1))$. We write $f(\lambda) \lesssim g(\lambda)$, if $f(\lambda) \leqslant g(\lambda) \cdot (1 + \mathrm{o}(1))$.

For a leakage function $f \colon F \to \{0, 1\}$, define the set $f^{-1}(b) := \{x \in F \colon f(x) = b\}$, where $b \in \{0, 1\}$.

For a finite field F, parameter $n \in \{2, 3, \dots\}$, and elements $\alpha_1, \alpha_2, \dots, \alpha_n \in F$, define the following *equivalence class*

$$[\alpha_1 \colon \alpha_2 \colon \cdots \colon \alpha_n] := \{(\Lambda \cdot \alpha_1, \Lambda \cdot \alpha_2, \dots, \Lambda \cdot \alpha_n) : \Lambda \in F^*\}$$

Supporting Material B shows that all elements in the same equivalence class have identical resilience/vulnerability to attacks.

3.1 Functions over Finite Fields

Let F be a prime field of order $p \geqslant 3$. This section defines some $F \to \mathbb{Z}$ functions.

$$|X|_p := \begin{cases} X', & \text{if } X = X' \mod p, X' \in \{0, 1, \dots, (p-1)/2\} \\ -X' & \text{if } X = X' \mod p, X' \in \{-(p-1)/2, \dots, -1\}. \end{cases} \quad (1)$$

$$\mathrm{sign}_p(X) := \begin{cases} +1, & \text{if } X \in \{0, 1, \dots, (p-1)/2\} \mod p \\ -1, & \text{if } X \in \{-(p-1)/2, \dots, -1\} \mod p. \end{cases} \quad (2)$$

We define the following quantity for $k, \ell, \Delta \in F$.

$$\Sigma_{k,\ell}^{(\Delta)} := \sum_{T \in F} \mathrm{sign}_p(kT) \cdot \mathrm{sign}_p(\ell(T - \Delta)). \quad (3)$$

Similarly, we define the following quantity for $k, \ell, m, \Delta, \Delta' \in F$.

$$\Sigma_{k,\ell,m}^{(\Delta,\Delta')} := \sum_{T \in F} \mathrm{sign}_p(kT) \cdot \mathrm{sign}_p(\ell(T-\Delta)) \cdot \mathrm{sign}_p(m(T-\Delta')). \tag{4}$$

We will require the following intermediate definitions for technical analysis: slight variations of the definitions above.

$$\widetilde{\mathrm{sign}}_p(X) := \begin{cases} +1, & \text{if } X \in \{1, \ldots, (p-1)/2\} \mod p \\ 0, & \text{if } X = 0 \mod p \\ -1, & \text{if } X \in \{-(p-1)/2, \ldots, -1\} \mod p. \end{cases} \tag{5}$$

$$\widetilde{\Sigma}_{k,\ell}^{(\Delta)} := \sum_{T \in F} \widetilde{\mathrm{sign}}_p(kT) \cdot \widetilde{\mathrm{sign}}_p(\ell(T-\Delta)) \tag{6}$$

$$\widetilde{\Sigma}_{k,\ell,m}^{(\Delta,\Delta')} := \sum_{T \in F} \widetilde{\mathrm{sign}}_p(kT) \cdot \widetilde{\mathrm{sign}}_p(\ell(T-\Delta)) \cdot \widetilde{\mathrm{sign}}_p(m(T-\Delta')). \tag{7}$$

3.2 Functions over Real Numbers

This section defines some $\mathbb{R} \to \mathbb{R}$ functions.

$$\mathrm{sign}(x) := \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases} \tag{8}$$

$$\varphi(x) := \mathrm{sign}\sin(2\pi x) \tag{9}$$

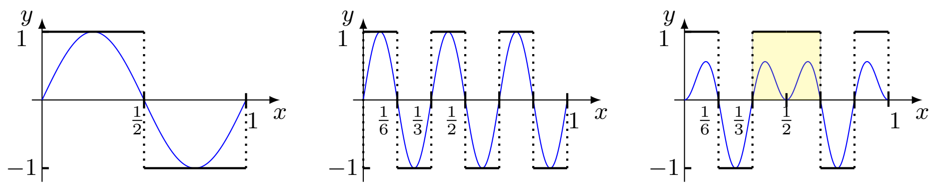
We define the following integral for $k, \ell \in \mathbb{Z}$ and $\delta \in \mathbb{R}$.

$$I_{k,\ell}^{(\delta)} := \int_0^1 \varphi(kt) \cdot \varphi(\ell(t-\delta))\, \mathrm{d}t. \tag{10}$$

Remark 7 (Intuition of the Square Waves). From standard Fourier analysis, it is well known that sine waves $\sin(2\pi \cdot x)$ and $\sin(2\pi \cdot 3x)$ are orthonormal, i.e., $\int_0^1 \sin(2\pi \cdot x) \cdot \sin(2\pi \cdot 3x)\, \mathrm{d}t = 0$. However, the square waves $\varphi(x) = \mathrm{sign}\sin(2\pi \cdot x)$ and $\varphi(3x) = \mathrm{sign}\sin(2\pi \cdot 3x)$ are not orthogonal (see Figure 3). Surprisingly, the square wave $\varphi(x)$ and the offset square wave $\varphi(3(x - 1/12))$ are orthogonal (see Figure 4). A technical objective of our work will be to determine the inner product $I_{k,\ell}^{(\delta)}$ between a square wave $\varphi(kx)$ and another offset square wave $\varphi(\ell(x-\delta))$. For example, we have $I_{1,3}^{(0)} = 1/3$ and $I_{1,3}^{(1/12)} = 0$.

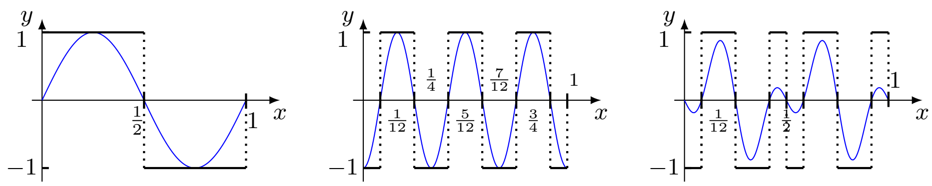
4 Security against Least Significant Bit Leakage

This section presents all results pertaining to the security of Shamir's secret-sharing scheme when $n = k = 2$. We begin with a powerful technical result that we prove.



$\varphi(x) = \operatorname{sign}\sin(2\pi x)$ $\quad$ $\varphi(3x) = \operatorname{sign}\sin(2\pi \cdot 3x)$ $\quad$ $\varphi(x) \cdot \varphi(3x)$

Fig. 3. Square waves $\varphi(x) = \operatorname{sign}\sin(2\pi \cdot x)$ and $\varphi(3x) = \operatorname{sign}\sin(2\pi \cdot 3x)$ are not orthogonal since $\int_0^1 \varphi(t) \cdot \varphi(3t)\,\mathrm{d}t = 1/3$. Blue lines draw the functions without the $\operatorname{sign}(\cdot)$ function.



$\varphi(x) = \operatorname{sign}\sin(2\pi x)$ $\quad$ $\varphi(3(x-1/12))$ $\quad$ $\varphi(x) \cdot \varphi(3(x-1/12))$

Fig. 4. Square waves $\varphi(x) = \operatorname{sign}\sin(2\pi \cdot x)$ and $\varphi(3(x-1/12)) = \operatorname{sign}\sin(2\pi \cdot 3(x - 1/12))$ are orthogonal since $\int_0^1 \varphi(t) \cdot \varphi(3(t-1/12))\,\mathrm{d}t = 0$. Blue lines draw the functions without the $\operatorname{sign}(\cdot)$ function.

Theorem 1 (Technical Result). *Consider the* $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ *secret-sharing scheme over a prime field* F_p*, where* $p \geqslant 3$*. For any* $(u, v) \in [\alpha_1 \colon \alpha_2]$*,*

$$\max_{s \in F} \mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0)) \;,\; \mathrm{LSB}(\mathsf{Share}(s))\right) = \begin{cases} \pm\frac{4(|u|_p + |v|_p) - (3/2)}{p}, & \text{if } |u|_p \cdot |v|_p / g^2 \text{ is even,} \\ \cos^2(\pi/2p) \cdot \frac{g^2}{|u|_p \cdot |v|_p} \quad \pm \frac{4(|u|_p + |v|_p) - (3/2)}{p} & \text{if } |u|_p \cdot |v|_p / g^2 \text{ is odd,} \end{cases}$$

where $g = \gcd\left(|u|_p, |v|_p\right)$*. Furthermore, for* $s = \pm(u^{-1} \cdot v - 1)^{-1} \in F^*$*, if* $\mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0)) \;,\; \mathrm{LSB}(\mathsf{Share}(s))\right) > \frac{4(|u|_p + |v|_p) - (3/2)}{p}$*, then there is an efficient distinguisher to distinguish the secret 0 and s with advantage at least*

$$\cos^2(\pi/2p) \cdot \frac{g^2}{|u|_p \cdot |v|_p} \quad - \frac{4(|u|_p + |v|_p) - (3/2)}{p}$$

using the LSB *leakage on the secret shares.*

Section 4.4 presents the proof outline for this result and Supporting Material C.6 presents the full proof. Using this theorem, we begin by stating and proving the corollaries mentioned in Section 1.2.

4.1 Statement and Proof of Corollary 1

Corollary 1. *Consider distinct evaluation places* $\vec{\alpha} = (\alpha_1, \alpha_2)$ *and the corresponding* $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ *secret-sharing scheme over the prime field* F_p*, where* $p \geqslant 3$*. Let* $(u, v) \in [\alpha_1 : \alpha_2]$ *such that* $|u|_p, |v|_p \leqslant B$*, where* $B = \lceil 8^{1/4} \cdot \sqrt{p} \rceil$*. Let* $g = \gcd(|u|_p, |v|_p)$*. Define*

$$\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha}) := \begin{cases} 0, & \text{if } |u|_p \cdot |v|_p / g^2 \text{ is even,} \\ \cos^2(\pi/2p) \cdot \frac{g^2}{|u|_p \cdot |v|_p}, & \text{if } |u|_p \cdot |v|_p / g^2 \text{ is odd.} \end{cases}$$

Then,

$$\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha}) = \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \quad \pm \left(\frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p} \right).$$

Proof. Use the LLL algorithm [31] to efficiently find $(u, v) \in [\alpha_1 : \alpha_2]$ with properties mentioned in the corollary (see Supporting Material A for details). Observe that the LHS of the expression in Theorem 1 is identical to $\varepsilon_{\mathrm{LSB}}(\vec{\alpha})$. From this observation, the corollary is immediate.

4.2 Statement and Proof of Corollary 2

Corollary 2. *Consider distinct evaluation places* $\vec{\alpha} = (\alpha_1, \alpha_2)$ *and the corresponding* $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ *secret-sharing scheme over the prime field* F_p*, where* $p \geqslant 3$*. Suppose the algorithm in Figure 2 determines* $\vec{\alpha}$ *to be secure. Then,*

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \leqslant \frac{1 + 8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Among all possible distinct evaluation places $\alpha_1, \alpha_2 \in F_p^*$*, the algorithm of Figure 2 determines at least*

$$\geqslant 1 \quad - \frac{\frac{1}{4} \cdot \sqrt{p} \cdot \ln p + \frac{3}{2} \cdot \sqrt{p} + \frac{1}{2}}{p - 2} \geqslant^{(*)} 1 \quad - \left(\frac{\ln p}{4\sqrt{p}} + \frac{5/2}{\sqrt{p}} \right).$$

fraction of them to be secure. The $(*)$ *inequality holds for any prime* $p \geqslant 11$*.*

Proof. **Proof of the first part.** The algorithm in Figure 2 declares $\vec{\alpha}$ to be secure either in Step 4 or Step 5.

Suppose our algorithm in Figure 2 declared that Shamir's secret-sharing scheme is secure in Step 4. In this case, $|u|_p \cdot |v|_p / g^2$ is even, where $g = \gcd(|u|_p, |v|_p)$. Using Corollary 1, we get that our estimation $\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})} = 0$. The relation between our estimation and insecurity yields

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \leqslant \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Suppose our algorithm in Figure 2 declared that Shamir's secret-sharing scheme is secure in Step 5. In this case, $|u|_p \cdot |v|_p / g^2 \geqslant \sqrt{p}$ and it is odd. Using Corollary 1, we get that our estimation $\varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}} \leqslant 1/\sqrt{p}$. The relation between our estimate and insecurity yields

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \leqslant \frac{1}{\sqrt{p}} + \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

This completes the proof of the first part of the corollary.

Proof of the second part. We prove that our algorithm outputs "may be insecure" only for an exponentially small fraction of the equivalence classes $[\alpha_1 \colon \alpha_2]$, for distinct evaluation places $\alpha_1, \alpha_2 \in F_p^*$.

First, observe that there are $(p-2)$ equivalence classes $[1\colon 2]$, $[1\colon 3]$, $\ldots$, $[1\colon (p-1)]$ (because $\alpha_1 \neq \alpha_2$ and $0 \notin \{\alpha_1, \alpha_2\}$).

Next, let us account for the instances when Figure 2 determines evaluation places $\vec{\alpha}$ may be insecure. Suppose $a = (u/g)$ and $b = (v/g)$, where $g = \gcd(u,v) \in \{1, 2, \ldots\}$. We need to upper bound the cardinality of the following set

$$S := \left\{ (a,b) \colon \text{odd } a,\ \text{odd } b,\ \text{and } |a \cdot b| \leqslant \sqrt{p} \right\}.$$

In this set, for any particular a, the corresponding positive b lies in the set $\{1, 3, 5, \ldots, 2n_a - 1\}$, such that $(2n_a - 1)$ is the largest odd number satisfying $a \cdot (2n_a - 1) \leqslant \sqrt{p}$. So, the number of potential odd positive b's is $n_a \leqslant (\sqrt{p} + a)/2a$. As a result, the total number of potential positive and negative candidates is at most $(\sqrt{p} + a)/a$. Let $(2s-1)$ be the largest odd number $\leqslant \sqrt{p}$. Therefore, we have

$$\begin{aligned}
\mathsf{card}(S) &\leqslant 2 \cdot \sum_{a \in \{1,3,\ldots,2s-1\}} \frac{\sqrt{p} + a}{a} = 2\sqrt{p}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2s-1}\right) + 2s \\
&\leqslant 2\sqrt{p} \cdot \left(1 + \int_1^s \frac{1}{2t-1}\,\mathrm{d}t\right) + (\sqrt{p} + 1) \\
&= \sqrt{p} \cdot \ln(2s-1) + 3\sqrt{p} + 1 \leqslant \frac{1}{2}\sqrt{p} \cdot \ln p + 3\sqrt{p} + 1.
\end{aligned}$$

Note that for every (a,b), we also counted $(-a,-b)$ in this set; both belong to the same equivalence class. So, every equivalence class is represented at least twice. Therefore, the number of equivalence classes for which our algorithm outputs "may be insecure" is $\leqslant \mathsf{card}(S)/2$. The fraction of equivalence classes that our algorithm declares "may be insecure" is

$$\leqslant \frac{\mathsf{card}(S)/2}{p-2} \leqslant \frac{\frac{1}{4} \cdot \sqrt{p} \cdot \ln p + \frac{3}{2} \cdot \sqrt{p} + \frac{1}{2}}{p-2}.$$

Asymptotically, the upper bound is $\lesssim \frac{1}{4} \cdot \frac{\ln p}{\sqrt{p}}$. Concretely, Supporting Material C.7 proves the upper bound

$$\leqslant \frac{\ln p}{4\sqrt{p}} + \frac{5/2}{\sqrt{p}},$$

for all $p \geqslant 11$.

4.3 Statement and Proof of Corollary 3

Corollary 3. *Consider distinct evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2)$ and the corresponding* $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ *secret-sharing scheme over the prime field F_p, where $p \geqslant 3$. If $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) > \frac{2 \cdot 8^{5/4}}{\sqrt{p}} + \frac{13}{p}$, then there is an efficient algorithm that generates $s \in F_p^*$ and can distinguish the secret 0 from the secret s with an advantage*

$$\geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \quad - \frac{2 \cdot 8^{5/4}}{\sqrt{p}} - \frac{13}{p}$$

by leaking the LSB *of the secret shares.*

Proof. Our efficient adversary outputs the s indicated in Theorem 1. After observing the leakage (ℓ_1, ℓ_2), this algorithm performs maximum likelihood decoding – computes whether secret 0 or secret s is more likely to have generated the observed leakage. Then, it predicts the most likely of the two events.

We emphasize that the secret $s' \in F^*$ that witnesses the maximum statistical distance between the leakage distributions $\mathrm{LSB}(\mathsf{Share}(0))$ and $\mathrm{LSB}(\mathsf{Share}(s'))$ may be different from the secret s defined above. Secret $s \in F^*$ witnesses the maximum *estimate* of the statistical distance between the distributions $\mathrm{LSB}(\mathsf{Share}(0))$ and $\mathrm{LSB}(\mathsf{Share}(s))$.

For brevity, define $\mathrm{err} := \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}$. Given $\vec{\alpha}$, we run the LLL algorithm [31] to obtain $(u, v) \in [\alpha_1 : \alpha_2]$ such that $|u|_p, |v|_p \leqslant B$, where $B = \lceil 8^{1/4} \cdot \sqrt{p} \rceil$. Define $g = \gcd(|u|_p, |v|_p)$.

We are given that $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) > 2 \cdot \mathrm{err}$. We claim that $\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha}) > \mathrm{err}$ and $|u|_p \cdot |v|_p / g^2$ is odd. Suppose not; then, there are two possibilities.

1. If $|u|_p \cdot |v|_p / g^2$ is even. In this case, $\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha}) = 0$ and, hence, $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \leqslant \mathrm{err}$, by Corollary 1; a contradiction.
2. If $\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha}) \leqslant \mathrm{err}$ and $|u|_p \cdot |v|_p / g^2$ is odd. In this case, $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \leqslant 2 \cdot \mathrm{err}$, by Corollary 1; a contradiction.

So, the signs of $\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha})$ and $\left(\frac{1}{p} \Sigma_{\alpha_1, \alpha_2}^{(0)} - \frac{1}{p} \cdot \Sigma_{\alpha_1, \alpha_2}^{(\Delta)}\right)$ are identical (by Claim 11). Using this property, Supporting Material C.8 proves that the advantage of the maximum likelihood decoder is

$$\geqslant \varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha}) - \mathrm{err} \geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) - 2 \cdot \mathrm{err}.$$

4.4 Proof outline of Theorem 1

For any $s \in F^*$, we prove Lemma 1 that obtains a closed-form estimate of

$$\mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right).$$

Then, we can solve for the optimal $s \in F^*$ that maximizes the statistical distance.

Lemma 1. *Consider the* $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ *secret-sharing scheme over a prime field* F_p*. For any secret* $s \in F_p^*$ *and* $(u, v) \in [\alpha_1 \colon \alpha_2]$,

$$\mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right) = \begin{cases} \pm\frac{4(|u|_p+|v|_p)-(3/2)}{p}, & \text{if } |u|_p \cdot |v|_p/g^2 \text{ is even} \\ \sin^2\left(|v|_p\pi \cdot \delta\right) \cdot \frac{g^2}{|u|_p \cdot |v|_p} \quad \pm\frac{4\left(|u|_p+|v|_p\right)-(3/2)}{p}, & \text{if } |u|_p \cdot |v|_p/g^2 \text{ is odd,} \end{cases}$$

where $g = \gcd\left(|u|_p, |v|_p\right)$, $\delta = \frac{\mathrm{sign}_p(\Delta)\cdot|\Delta|_p}{p} \in \mathbb{Q}$, *and* $\Delta = \left(s \cdot 2^{-1}\right)\cdot\left(u^{-1} - v^{-1}\right) \in F_p^*$.

Supporting Material C.5 proves Lemma 1. Below, we present a high-level overview of the proof.

Step 1. Using a combinatorial argument, we connect the statistical distance between the leakages to the difference between two sums of oscillatory functions.

Lemma 2. *Consider the* $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ *secret-sharing scheme over a prime field* F_p*. For any secret* $s \in F_p$ *and* $(u, v) \in [\alpha_1 \colon \alpha_2]$,

$$\mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right) = \frac{1}{2p} \cdot \left|\Sigma_{u,v}^{(0)} - \Sigma_{u,v}^{(\Delta)}\right|,$$

where $\Delta := \left(s \cdot 2^{-1}\right) \cdot \left(u^{-1} - v^{-1}\right)$*, a linear automorphism over* F_p*.*

Supporting Material C.1 proves Lemma 2.

Step 2. Recall that $\mathrm{sign}_p(X = 0) = +1$ and $\mathrm{sign}(x = 0) = 0$. Due to this mismatch, we defined an intermediate function $\widetilde{\mathrm{sign}}_p(X = 0) = 0$. So, our next objective is to relate the quantities $\Sigma_{k,\ell}^{(\Delta)}$ with $\widetilde{\Sigma}_{k,\ell}^{(\Delta)}$.

Lemma 3. *For any* $k, \ell, \Delta \in F_p$,

$$\Sigma_{k,\ell}^{(\Delta)} = \widetilde{\Sigma}_{k,\ell}^{(\Delta)} \quad + \left(\sum_{T\in\{0,\Delta\}} \mathrm{sign}_p(kT) \cdot \mathrm{sign}_p(\ell(T - \Delta))\right).$$

Supporting Material C.2 proves Lemma 3.

Step 3. Next, our objective is to estimate the sum $\frac{1}{p} \cdot \widetilde{\Sigma}_{k,\ell}^{(\Delta)}$ using the integral $I_{k,\ell}^{(\delta)}$, for an appropriately define $\delta \in \mathbb{R}$.

Lemma 4. *For any $k, \ell, \Delta \in F_p$,*

$$\frac{1}{p} \cdot \widetilde{\Sigma}_{k,\ell}^{(\Delta)} = \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot I_{|k|_p, |\ell|_p}^{(\delta)} \quad \pm \frac{4(|k|_p + |\ell|_p) - 2}{p},$$

where $\delta = \frac{\mathrm{sign}_p(\Delta) \cdot |\Delta|_p}{p} \in \mathbb{Q}$.

Supporting Material C.3 proves Lemma 4.

Step 3. Finally, we compute the value of the integral $I_{k,\ell}^{(\delta)}$.

Lemma 5. *For any $k, \ell \in \{1, 2, \dots\}$ and $\delta \in \mathbb{R}$,*

$$I_{k,\ell}^{(\delta)} = \begin{cases} 0, & \text{if } k \cdot \ell / g^2 \text{ is even} \\ \\ \cos(2\ell\pi \cdot \delta) \cdot \frac{g^2}{k\ell}, & \text{if } k \cdot \ell / g^2 \text{ is odd.} \end{cases},$$

where $g = \gcd(k, \ell)$.

Supporting Material C.4 proves Lemma 5. Intuitively, if the highest power of 2 dividing k is different from the highest power of 2 dividing ℓ, then $k\ell/g^2$ is even and $I_{k,\ell}^{(\delta)} = 0$. If the highest power of 2 dividing k is identical to the highest power of 2 dividing ℓ, then $k\ell/g^2$ is odd and $I_{k,\ell}^{(\delta)} \neq 0$.

Step 4. Sequentially performing the substitutions above, we can estimate the statistical distance using the integrals, which yields Lemma 1.

Efficient distinguisher construction. We present an efficient maximum likelihood distinguisher in Supporting Material C.6.

5 Security against all Physical Bit Leakage

We consider $\mathsf{ShamirSS}(n = 2, k = 2, (\alpha_1, \alpha_2))$ over the prime field F of order $p \geqslant 3$. This section considers p a Mersenne prime, i.e., $p = 2^\lambda - 1$, where λ is the security parameter. Some initial Mersenne primes are $3, 7, 31, 127, 8191$, and 131071. The largest Mersenne prime, currently known, is $2^{82,589,933} - 1$.

5.1 Properties of Mersenne Primes

Mersenne primes have fascinating properties.

Proposition 1. *Let F be a prime field of order $p = 2^\lambda - 1$. Suppose $x \in F$ and define $x' = x \cdot (2^i) \in F$, where $i \in \{-\lambda + 1, \dots, 0, 1, \dots, \lambda - 1\}$. Then the binary representation of x' is a cyclic left rotation of the binary representation of x by i bits.*

We clarify that if i is negative, then "i bit cyclic left rotation" is the same as "$|i|$ bit cyclic right rotation." This proposition is straightforward from the identity that $2^\lambda = 1 \mod p$. Additionally, it implies that $2^{\lambda+i} = 2^i \mod p$, for all negative $i \in \{-p+1, \ldots, -1\}$.

Suppose $i \in \{0, 1, \ldots, \lambda - 1\}$. Let $E_i \subseteq \{0, 1, \ldots, p-1\} = F$ be the set of all element $x \in F$ such that the binary representation of x has 0 at its i-th position. We remind the reader that $i = 0$ indicates the least significant bit, and $i = (\lambda - 1)$ indicates the most significant bit. We represent $E = \{0, 2, \ldots, (p-1)/2\}$ as the set of all "even elements" in F.

Proposition 2. *For all $i \in \{0, 1, \ldots, \lambda - 1\}$, we have $E_i = E \cdot (2^i)$.*

Note that if $x \in E_i$ then $x \cdot (2^{-i}) \in E$. Moreover, for any $x' \in E$, we have $x \cdot (2^i) \in E_i$. Both these properties hold due to Proposition 1.

5.2 Leakage Resilience to Physical Bit Leakage

Let $\mathrm{LSB}_i \colon F \to \{0, 1\}$ represent the function that outputs the i-th least significant bit in the binary representation. So, for example, $\mathrm{LSB}_i^{-1}(0) = E_i = E \cdot (2^i)$. We aim to investigate the leakage resilience of $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ over the prime field F with order $p = 2^\lambda - 1$ against physical bit leakage attacks. Consider a leakage attack that leaks the i-th LSB of the first secret share and the j-th LSB of the second secret share. We represent this leakage as $\vec{\mathrm{LSB}}_{i,j}$.

Leakage attack when $2^k\alpha_1 = \alpha_2$. Although $\alpha_1 \neq \alpha_2$, it may be possible that $2^k\alpha_1 = \alpha_2$, for some $k \in \{0, 1, \ldots, \lambda - 1\}$. We prove that the secret-sharing scheme is insecure, taking care of this case in the algorithm of Figure 1.

Since α_1 and α_2 are distinct, it must be the case that $2^k\alpha_1 = \alpha_2$, where $k \in \{1, 2, \ldots, \lambda - 1\}$. Suppose we are leaking the i-th bit of the first secret share and the j-th bit of the second secret share, such that $j - i = k$.

Suppose the secret is $s \in F$. Then, the secret share at evaluation place X is $s + uX$, for uniformly random $u \in F$. The joint distribution of leakage is

$$(\mathrm{LSB}_i(s + u\alpha_1), \mathrm{LSB}_j(s + u\alpha_2)).$$

Since $E_i = E2^i$ and $E_j = E2^j$, this joint distribution is identical to

$$\left(\mathrm{LSB}_0(s2^{-i} + u\alpha_1 2^{-i}), \mathrm{LSB}_0(s2^{-j} + u\alpha_2 2^{-j})\right).$$

Define some variable renaming. Let $v := u2^{-j}$ and $t := s2^{-j}$. The joint distribution of leakage is (for uniformly random $v \in F$)

$$\left(\mathrm{LSB}_0(t2^k + v\alpha_1 2^k), \mathrm{LSB}_0(t + v\alpha_2)\right) \equiv \left(\mathrm{LSB}_0(t2^k + v\alpha_2), \mathrm{LSB}_0(t + v\alpha_2)\right),$$

because $2^k\alpha_1 = \alpha_2$.

For $t = 0$, both the leakage bits are identical. On the other hand, for $t = t^* := (2^k - 1)^{-1}$, the joint distribution of leakage is

$$(\mathrm{LSB}_0(1 + t^* + v\alpha_2), \mathrm{LSB}_0(t^* + v\alpha_2))$$

These two leakage bits are different with $(1 - 1/p)$ probability. Therefore, one can distinguish the secret 0 and secret $t^* 2^j$ with $(1 - 1/p) \sim 1$ advantage by leaking $\vec{\text{LSB}}_{i,j}$; whence the following lemma follows.

Lemma 6. *Let F_p be the prime field of order $p = 2^\lambda - 1$. Consider distinct evaluation places $\alpha_1, \alpha_2 \in F_p^*$ such that $2^k \cdot \alpha_1 = \alpha_2$ for some $k \in \{0, 1, \ldots, \lambda - 1\}$. Then,*

$$\mathsf{SD}\left(\vec{\text{LSB}}_{i,j}(\mathsf{Share}(0))\ ,\ \vec{\text{LSB}}_{i,j}(\mathsf{Share}(s))\right) \geqslant 1 - \frac{1}{p},$$

where $i, j \in \{0, 1, \ldots, \lambda - 1\}$, $j - i = k \mod \lambda$, and $s = (2^k - 1)^{-1} \cdot 2^j$.

Reduction to the LSB Attack. Due to the properties of the F_p, where p is a Mersenne prime, we can reduce arbitrary physical bit attacks on $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ to LSB leakage attacks on $\mathsf{ShamirSS}(2, 2, \vec{\alpha'})$, for an appropriately defined $\vec{\alpha'}$.

Lemma 7. *Let F_p be a prime field of order $p = 2^\lambda - 1$. Consider evaluation places $\alpha_1, \alpha_2 \in F_p^*$ such that $2^k \cdot \alpha_1 \neq \alpha_2$, for all $k \in \{0, 1, \ldots, \lambda - 1\}$. Consider the leakage attack $\vec{\text{LSB}}_{i,j}$ for any $i, j \in \{0, 1, \ldots, \lambda - 1\}$. Define $\alpha'_1 = 2^{-i} \cdot \alpha_1$ and $\alpha'_2 = 2^{-j} \cdot \alpha_2$. For any $s \in F_p$, let D denote the joint leakage distribution generated by the leakage function $\vec{\text{LSB}}_{i,j}$ when the secret shares are generated using the $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ secret-sharing scheme. Likewise, D' denotes the joint leakage distribution generated by the leakage function $\vec{\text{LSB}}$ when the secret shares are generated using the $\mathsf{ShamirSS}(2, 2, \vec{\alpha'})$ secret-sharing scheme instead. Then, the distributions D and D' are identical.*

Since $2^k \cdot \alpha_1 \neq \alpha_2$, for all $k \in \{0, 1, \ldots, \lambda - 1\}$, we conclude that $\alpha'_1 \neq \alpha'_2$, for all $i, j \in \{0, 1, \ldots, \lambda - 1\}$. Therefore, the secret-sharing scheme $\mathsf{ShamirSS}(2, 2, \vec{\alpha'})$ is valid. Supporting Material C.9 proves that the distributions D and D' are identical, for all $s \in F_p$, using Proposition 2.

5.3 Statement and Proof of Corollary 4

Corollary 4. *Let F_p be the prime field of order $p = 2^\lambda - 1$. Consider distinct evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2)$ and the corresponding secret-sharing scheme $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$. Define*

$$\varepsilon^{(\text{OUR})}_{\text{PHYS}} = \begin{cases} 1, & \text{if } 2^t \cdot \alpha_1 = \alpha_2 \\ & \text{for some } t \in \{0, 1, \ldots, \lambda - 1\}, \\ \max\limits_{k \in \{0,1,\ldots,p-1\}} \varepsilon^{(\text{OUR})}_{\text{LSB}}\left(\ (2^k \alpha_1, \alpha_2)\ \right), & \text{if } 2^t \cdot \alpha_1 = \alpha_2 \\ & \text{for all } t \in \{0, 1, \ldots, \lambda - 1\}. \end{cases}$$

Then,

$$\varepsilon^{(\text{OUR})}_{\text{PHYS}}(\vec{\alpha}) = \varepsilon_{\text{PHYS}}(\vec{\alpha}) \quad \pm \left(\frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}\right).$$

Proof. If $2^t \cdot \alpha_1 = \alpha_2$, for some $t \in \{0, 1, \ldots, \lambda - 1\}$, we have $\varepsilon^{(\text{OUR})}_{\text{PHYS}}(\vec{\alpha}) = 1$. Lemma 6 presents a physical bit leakage attack with distinguishing advantage $1 - 1/p$; therefore, $\varepsilon_{\text{PHYS}}(\vec{\alpha}) \geqslant 1 - 1/p$. So, we conclude that

$$\varepsilon^{(\text{OUR})}_{\text{PHYS}}(\vec{\alpha}) = \varepsilon_{\text{PHYS}}(\vec{\alpha}) \pm \frac{1}{p}.$$

If $2^t\alpha_1 \neq \alpha_2$, for all $t \in \{0, 1, \ldots, \lambda - 1\}$, Lemma 7 shows that the leakage distribution of $\text{LSB}_{i,j}$ on $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ is identical to the leakage distribution $\vec{\text{LSB}}$ on $\mathsf{ShamirSS}(2, 2,\ (2^{-i}\alpha_1, 2^{-j}\alpha_j)\ )$. Recall that the secret-sharing scheme $\mathsf{ShamirSS}(2, 2,\ (2^{-i}\alpha_1, 2^{-j}\alpha_j)\ )$ is identical to the secret-sharing scheme $\mathsf{ShamirSS}(2, 2,\ (2^{j-i}\alpha_1, \alpha_j)\ )$, by Lemma 10 in Supporting Material B. Therefore, we conclude the following:

$$\varepsilon_{\text{PHYS}}(\vec{\alpha}) = \max_{k \in \{0,1,\ldots\}} \varepsilon_{\text{LSB}}\left(\ (2^k\alpha_1, \alpha_2)\ \right).$$

We know that our estimation $\varepsilon^{(\text{OUR})}_{\text{LSB}}(\cdot)$ is a tight estimation of $\varepsilon_{\text{LSB}}(\cdot)$, by Corollary 1. Therefore, we conclude that

$$\varepsilon^{(\text{OUR})}_{\text{PHYS}}(\vec{\alpha}) = \varepsilon_{\text{PHYS}}(\vec{\alpha}) \quad \pm \left(\frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}\right).$$

5.4 Statement and Proof of Corollary 5

Corollary 5. *Let F_p be the prime field of order $p = 2^\lambda - 1$. Consider distinct evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2)$ and the corresponding $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ secret-sharing scheme over the prime field F_p. Suppose the algorithm in Figure 1 determines $\vec{\alpha}$ to be secure. Then,*

$$\varepsilon_{\text{PHYS}}(\vec{\alpha}) \leqslant \frac{1 + 8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Among all possible distinct evaluation places $\alpha_1, \alpha_2 \in F_p^*$, the algorithm of Figure 2 determines at least

$$\geqslant 1 \quad - \frac{\ln p}{\ln 2} \cdot \frac{\frac{1}{4} \cdot \sqrt{p} \cdot \ln p + \frac{3}{2} \cdot \sqrt{p} + \frac{1}{2}}{p - 2} \geqslant^{(*)} 1 \quad - \frac{\ln p}{\ln 2} \cdot \left(\frac{\ln p}{4\sqrt{p}} + \frac{5/2}{\sqrt{p}}\right).$$

fraction of them to be secure. The $(*)$ inequality holds for all $p \geqslant 11$.

Proof. **Proof of the first part.** If the algorithm in Figure 1 determined (α_1, α_2) to be secure, then the algorithm in Figure 2 determined $(2^k\alpha_1, \alpha_2)$ to be secure, for all $k \in \{0, 1, \ldots, \lambda - 1\}$. For $k \in \{0, 1, \ldots, \lambda - 1\}$, by Corollary 2, we get the bound that

$$\varepsilon_{\text{LSB}}\left(\ (2^k\alpha_1, \alpha_2)\ \right) \leqslant \frac{1 + 8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Just like the proof of Corollary 4, we have

$$\varepsilon_{\text{PHYS}}(\vec{\alpha}) = \max_{k\in\{0,1,\ldots,\lambda-1\}} \varepsilon_{\text{LSB}}\left(\ (2^k\alpha_1,\alpha_2)\ \right) \leqslant \frac{1+8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Proof of the second part. If the algorithm in Figure 1 outputs "may be insecure" then there is some $k \in \{0,1,\ldots,\lambda-1\}$ such that the algorithm in Figure 2 outputs "may be insecure" for $(2^k\alpha_1,\alpha_2)$. Corollary 2 proves that the algorithm in Figure 2 outputs "may be insecure" for at most

$$\frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p + \frac{3}{2}\cdot\sqrt{p}+\frac{1}{2}}{p-2}$$

fraction of the equivalence classes. By, a union bound over $k \in \{0,1,\ldots,\lambda-1\}$, Figure 1 outputs "may be insecure" for at most

$$\log_2 p\cdot\frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p + \frac{3}{2}\cdot\sqrt{p}+\frac{1}{2}}{p-2}$$

fraction of the equivalence classes.

5.5 Statement and Proof of Corollary 6

Corollary 6. *Let F_p be the prime field with order $p = 2^\lambda - 1$. Consider distinct evaluation places $\vec{\alpha} = (\alpha_1,\alpha_2)$ and the corresponding* $\mathsf{ShamirSS}(2,2,\vec{\alpha})$ *over F_p. If $\varepsilon_{\text{PHYS}}(\vec{\alpha}) > \frac{2\cdot 8^{5/4}}{\sqrt{p}} + \frac{13}{p}$, then there is an efficient algorithm that generates $(s,f) \in F_p^* \times \text{PHYS}$ and can distinguish the secret 0 from the secret s with an advantage*

$$\geqslant \varepsilon_{\text{PHYS}}(\vec{\alpha}) \quad - \frac{2\cdot 8^{5/4}}{\sqrt{p}} - \frac{13}{p}$$

by leaking f from the secret shares.

Proof. If there is $t \in \{0,1,\ldots,\lambda-1\}$ such that $2^t\alpha_1 = \alpha_2$, then Lemma 6 presents an explicit leakage attack that suffices for this corollary.

If there $2^t\alpha_1 \neq \alpha_2$ for all $t \in \{0,1,\ldots,\lambda-1\}$, then Lemma 7 helps relate physical bit attacks and LSB attacks. Suppose k is the witness such that $\varepsilon^{(\text{OUR})}_{\text{PHYS}}(\vec{\alpha}) = \varepsilon^{(\text{OUR})}_{\text{LSB}}\left(\ (2^k\alpha_1,\alpha_2)\ \right)$. Then, consider the adversary against $\mathsf{ShamirSS}(2,2,\ (2^k\alpha_1,\alpha_2)\ ;)$ that uses the LSB attack as guaranteed by Corollary 3. Lemma 7 proves that the leakage distribution of the physical bit attack $\vec{\text{LSB}}_{0,k}$ on $\mathsf{ShamirSS}(2,2,\vec{\alpha})$ secret-sharing scheme has an identical distribution. So, we run the adversary of Corollary 3 by leaking $\vec{\text{LSB}}_{0,k}$ from the secret shares of the $\mathsf{ShamirSS}(2,2,\vec{\alpha})$ secret-sharing scheme.

5.6 Statement and Proof of Corollary 7

Corollary 7. *Let F_p be the prime field of order $p = 2^\lambda - 1$. Define $t := \lfloor \lambda/2 \rfloor$. Consider $\vec{\alpha} = (\alpha_1, \alpha_2) \in [1\colon 2^t - 1]$. Then*

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{4 \cdot \left(2^{\lfloor \lambda/2 \rfloor} + 2^{\lceil \lambda/2 \rceil}\right) - 6}{p}.$$

Proof. For the proof, fix $\alpha_1 = 1$ and $\alpha_2 = 2^{\lfloor \lambda/2 \rfloor} - 1$. We shall compute $\varepsilon_{\mathrm{LSB}}(2^i \cdot \alpha_1, \alpha_2)$ for all $i \in \{0, 1, \ldots, \lambda - 1\}$ using Lemma 1. The bound in our corollary will be the maximum of these individual upper bounds on $\varepsilon_{\mathrm{LSB}}(\cdot)$.

Case A: $i = 0$**.** We are interested in computing the security of the evaluation places $(2^i \alpha_1, \alpha_2)$ We use $(u, v) = (1, 2^t - 1)$, where $t = \lfloor \lambda/2 \rfloor$. Note that u, v are relatively prime and $|u|_p = 1$ and $|v|_p = 2^t - 1$. Both these evaluation places are odd. Therefore, by Lemma 1, we have

$$\varepsilon_{\mathrm{LSB}}(2^i \cdot \alpha_1, \alpha_2) \leqslant \frac{1}{2^t - 1} + \frac{4 + 4 \cdot (2^t - 1) - 2}{p}.$$

Case B: $1 \leqslant i \leqslant \lfloor \lambda/2 \rfloor$**.** We are interested in the security of $(u, v) = (2^i, 2^t - 1)$, where $t = \lfloor \lambda/2 \rfloor$. Note that u and v are relatively prime, u is even, and v is odd. Therefore, by Lemma 1, we have

$$\varepsilon_{\mathrm{LSB}}(2^i \cdot \alpha_1, \alpha_2) \leqslant \frac{4 \cdot 2^i + 4 \cdot (2^t - 1) - 2}{p}.$$

Case C: $\lfloor \lambda/2 \rfloor + 1 \leqslant i \leqslant \lambda - 1$**.** We are interested in the security of $(u, v) = (2^i, 2^t - 1)$, where $t = \lfloor \lambda/2 \rfloor$. Note that $t + 1 \leqslant i \leqslant \lambda - 1$. Define $(u', v') := 2^{\lambda - t} \cdot (u, v) \in [u\colon v]$. Observe that

$$\begin{aligned} u' &= 2^{\lambda - t} \cdot u \mod 2^\lambda - 1 = 2^{i - t} \\ v' &= 2^{\lambda - t} \cdot v \mod 2^\lambda - 1 = -(2^{\lambda - t} - 1). \end{aligned}$$

Substitute $u' = 2^j$, where $1 \leqslant j \leqslant \lfloor \lambda/2 \rfloor$, and $v' = -(2^{\lambda - t} - 1)$. Therefore, by Lemma 1, we have

$$\varepsilon_{\mathrm{LSB}}(2^i \cdot \alpha_1, \alpha_2) \leqslant \frac{4 \cdot 2^j + 4 \cdot (2^{\lambda - t} - 1) - 2}{p}.$$

Supporting Material C.10 proves the following upper bound on the insecurity for all $0 \leqslant i < \lambda$.

$$\varepsilon_{\mathrm{LSB}}(2^i \cdot \alpha_1, \alpha_2) \leqslant \frac{4 \cdot (2^{\lfloor \lambda/2 \rfloor} + 2^{\lceil \lambda/2 \rceil}) - 6}{p}$$

6 Extension to arbitrary Number of Parties

We extend our derandomization results to Shamir's secret-sharing scheme with the reconstruction threshold k equal to the number of parties $n \in \{2, 3, \dots\}$. We prove the following general lifting theorem. Corollary 8 is a consequence of this theorem.

Theorem 2. *Consider* $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ *over a prime field* F*. For every* $i \in \{1, 2, \dots, n\}$*, define* $\beta_i := \left(\alpha_i \prod_{j \neq i}(\alpha_i - \alpha_j)\right)^{-1}$*. Suppose there are two indices* $1 \leqslant i^* < j^* \leqslant n$ *such that* $\mathsf{ShamirSS}(2, 2, (\beta_{i^*}, \beta_{j^*}))$ *has* ε*-insecurity against physical bit leakages. Then,* $\mathsf{ShamirSS}(n, n, (\alpha_1, \alpha_2, \dots, \alpha_n))$ *has at most* 2ε*-insecurity against physical bit leakages.*

The proof of this theorem is Fourier-analytic and uses properties of the Generalized Reed-Solomon codes.

Generalized Reed-Solomon Code. A generalized Reed-Solomon code over a prime field F with message length k and block length n consists of an encoding function $\mathsf{Enc}\colon F^k \to F^n$ and decoding function $\mathsf{Dec}\colon F^n \to F^k$. It is specified by distinct *evaluation places* $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ and a *scaling vector* $\vec{u}$ such that for all $1 \leqslant i \leqslant n$, $u_i \in F^*$. Given $\vec{\alpha}$ and $\vec{u}$, the *encoding function* is

$$\mathsf{Enc}(m_1, \dots, m_k) := \left(u_1 \cdot f(\alpha_1), \dots, u_n \cdot f(\alpha_n)\right),$$

where $f(X) := m_1 + m_2 X + \cdots + m_k X^{k-1}$. We represent this code as $[n, k, \vec{\alpha}, \vec{u}]_F$-GRS.

The following standard properties of generalized Reed-Solomon codes shall be helpful for our extension to an arbitrary number of parties [20, 32].

Imported Theorem 1 (Properties of GRS) *The dual code of* $[n, k, \vec{\alpha}, \vec{u}]_F$*-GRS is identical to the* $[n, n-k, \vec{\alpha}, \vec{v}]_F$*-GRS, where for all* $1 \leqslant i \leqslant n$*,*

$$v_i^{-1} := u_i \prod_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_i - \alpha_j).$$

In particular, when $n = k$*, the dual code is the set* $\{\beta \cdot (v_1, v_2, \dots, v_n) \colon \beta \in F\}$*, a dimension one vector space over* F*.*

We will apply this theorem to the dual of the code containing all possible secret shares of the secret 0 in $[n, n, \vec{\alpha}]$-Shamir secret-sharing.

Since the proof of Theorem 2 is entirely Fourier-analytic, it is presented in Supporting Material D along with a brisk introduction to (elementary) Fourier analysis.

6.1 Statement and Proof of Corollary 8

Corollary 8. *Let F_p be the prime field of order $p = 2^\lambda - 1$. Fix any $n \in \{3, 4, \dots\}$. There is a probabilistic efficient algorithm to choose distinct evaluation places $\vec{\alpha}$ such that*

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{2 \cdot 8^{5/4}}{\sqrt{p}} + \frac{13}{p}.$$

The failure probability of this algorithm is

$$\leqslant \frac{n+1}{p} + \left(\frac{1}{4 \ln 2} \cdot \frac{(\ln p)^2}{\sqrt{p}} + \frac{5}{2 \ln 2} \cdot \frac{\ln p}{\sqrt{p}} \right).$$

Proof. Choose arbitrary distinct evaluation places $\alpha_1, \alpha_2, \alpha_4, \dots, \alpha_n \in F_p^*$. Choose α_3 uniformly at random from the set $F_p \setminus \{\alpha_1\}$. The probability that the evaluation places $(\alpha_1, \alpha_2, \dots, \alpha_n)$ are *not* all distinct is

$$\leqslant \frac{n-2}{p}.$$

Define $\beta_i := \left(\alpha_i (\prod_{j \neq i} (\alpha_i - \alpha_j) \right)^{-1}$ as in Theorem 2, for $i \in \{1, \dots, n\}$. Observe that choosing $\vec{\alpha}$ at random does not necessarily imply that $\vec{\beta}$ is uniformly and independently random over F_p. For this paper, we will prove a result that is easy to prove and sufficient for our context.

Lemma 8. *For $n \geqslant 3$, the distribution of the equivalence class $[\beta_1 \colon \beta_2]$ is $2/(p-1)$-close to the uniform distribution over the equivalence classes $[1:2], [1:3], \dots, [1:p-1]$, for*

1. *Arbitrary $\alpha_1, \alpha_2 \in F_p^*$ such that $\alpha_1 \neq \alpha_2$,*
2. *Arbitrary $\alpha_4, \dots, \alpha_n$ satisfying $\{\alpha_1, \alpha_2\} \cap \{\alpha_4, \dots, \alpha_n\} = \emptyset$, and*
3. *The evaluation place α_3 is chosen uniformly at random from the set $F_p \setminus \{\alpha_1\}$.*

Supporting Material F.1 proves this lemma. We use the algorithm in Figure 1 to test whether evaluation places in the equivalence class $[\beta_1 \colon \beta_2]$ is ε-secure, where

$$\varepsilon \leqslant \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

This guarantee is from Corollary 5. The probability of the algorithm in Figure 1 to return "may be insecure" is also exponentially small

$$\leqslant \left(\frac{1}{4 \ln 2} \cdot \frac{(\ln p)^2}{\sqrt{p}} + \frac{5}{2 \ln 2} \cdot \frac{\ln p}{\sqrt{p}} \right)$$

(again by Corollary 5). If no such pair of secure indices exit, then report *failure*. Otherwise, if one such pair exists, by Theorem 2, $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ has insecurity

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant 2\varepsilon.$$

By union bound, the failure probability is

$$\begin{aligned}
&\leqslant \frac{n-2}{p} + \frac{2}{p-1} + \left(\frac{1}{4\ln 2}\cdot\frac{(\ln p)^2}{\sqrt{p}} + \frac{5}{2\ln 2}\cdot\frac{\ln p}{\sqrt{p}}\right) \\
&\leqslant \frac{n+1}{p} + \left(\frac{1}{4\ln 2}\cdot\frac{(\ln p)^2}{\sqrt{p}} + \frac{5}{2\ln 2}\cdot\frac{\ln p}{\sqrt{p}}\right). \qquad \text{(for } p \geqslant 3\text{)}
\end{aligned}$$

One can boost the success probability exponentially by repeating this experiment.

7 The Case of $(n, k) = (3, 2)$

Lemma 9. *Let F_p be a prime field of order $p = 2^\lambda - 1$. Consider distinct evaluation places $(\alpha_1, \alpha_2, \alpha_3)$. Let $\varepsilon(\vec{\alpha})$ denote the insecurity of the* ShamirSS$(3, 2, \vec{\alpha})$ *secret-sharing scheme against physical bit leakage attacks. For $1 \leqslant i < j \leqslant 3$, denote the insecurity of the* ShamirSS$(2, 2, (\alpha_i, \alpha_j))$ *secret-sharing scheme against physical bit leakage attacks by $\varepsilon_{\mathrm{PHYS}}(\alpha_i, \alpha_j)$. Then,*

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \sum_{1 \leqslant i < j \leqslant 3} \varepsilon_{\mathrm{PHYS}}(\alpha_i, \alpha_j) + \frac{1}{p}.$$

Supporting Material E presents the full proof of this lemma. Note that if $\varepsilon_{\mathrm{PHYS}}(\alpha_i, \alpha_j)$ is large, then there is a leakage attack on ShamirSS$(3, 2, \vec{\alpha})$.

8 Prior Related Works

Local leakage resilience. Several works have constructed new secret-sharing schemes that are resilient to leakage attacks [7, 2, 45, 3, 29, 8, 15, 16, 23, 13, 38, 11]. There is a significant interest in characterizing the leakage-resilience of practical secret-sharing schemes, like the additive and Shamir's secret-sharing scheme. [33] proved that, for reconstruction threshold $k = 2$ and an arbitrary number of parties n, choosing evaluation places at random yields a leakage-resilient Shamir secret-sharing scheme with high probability against physical bit leakage. A sequence of works also determined the optimal leakage attack [33, 1, 35]. Other Monte Carlo constructions have also been proposed in [37, 34].

Another flavor of results characterizes the leakage-resilience of Shamir's secret-sharing scheme for a large number of parties. For example, when $k \geqslant 0.78n$, Shamir's secret-sharing scheme (with any evaluation places) is leakage-resilient to (arbitrary) one-bit local leakage. Here the insecurity is exponentially small in n [5, 6, 37, 36]. Contrast this with our scenario, where the insecurity is exponentially small in the security parameter, which is independent of n. [40] proved that Shamir's secret-sharing scheme is insecure to local leakage when n/k is large.

Square wave function families. Various families of square waves find wide applications in science and engineering. For example, consider the ones proposed by Haar [19], Walsh [48], and Rademacher [41]. In our work, we connect the leakage resilience of secret-sharing schemes with the properties of another family of square waves (see for example [47, 22, 21])

$$\left\{ \operatorname{sign}\sin(2\pi k \cdot x) \right\}_{k\in\mathbb{Z}}.$$

Previous works [47, 22] have studied the orthogonality of this family of waves. Our objective is to study, more generally, the "similarity" among these waves and their offsets – functions of the form $\operatorname{sign}\sin(2\pi k \cdot (x-\delta))$, for $\delta \in \mathbb{R}$. Zero similarity, in our context, coincides with orthogonality.

Simultaneous Diophantine Approximation. Solving simultaneous Diophantine approximation problems is a well-studied problem. This problem arises when choosing a "good basis" for a lattice. In our context, for an odd prime p, given distinct $\alpha_1, \alpha_2 \in \{1, 2, \dots, p-1\}$, our objective is to find $q \in \{1, 2, \dots, p-1\}$ such that $q\alpha_1 \mod p$ and $q\alpha_2 \mod p$ are either in the range $\{1, \dots, \sqrt{p}\}$ or $\{p-\sqrt{p}, \dots, p-1\}$. The integers $q\alpha_1 \mod p$ and $q\alpha_2 \mod p$, intuitively, have "small norm $\mod p$." We will use the classical LLL algorithm [31] to efficiently achieve this objective (see Supporting Material A).

The Dirichlet approximation theorem [43, 44] states that, for any $\alpha \in \mathbb{R}^d$ and any positive integer N, there is a denominator $1 \leqslant q \leqslant N^d$ such that

$$\max_{i\in\{1,2,\dots,d\}} \{q\alpha_i\} \leqslant \frac{1}{N}.$$

Computing this solution is computationally challenging [30]. However, we can efficiently solve this problem by slightly weakening the upper bound on q. The seminal LLL algorithm [31], in particular, for $\alpha \in \mathbb{Q}^d$, finds $1 \leqslant q \leqslant 2^{d(d+1)/4} \cdot N^d$ such that

$$\max_{i\in\{1,2,\dots,d\}} \{q\alpha_i\} \leqslant \frac{1}{N}.$$

9 Open Problems and Technical Bottlenecks

Motivated by applications in leakage-resilient MPC [25], there is a significant interest in designing explicit secret-sharing schemes for various access structures (for example, threshold and Q_2 access structures) that are resilient to leakage attacks. Our results contribute to this general research area. Below, we highlight some immediate extensions of our work and their technical challenges.

Investigating the case of composite order fields for our problem is open. Even the probabilistic version of this problem is open for composite order fields – [33] proved their probabilistic result *only for prime fields*. Naïvely interpreting F_{p^t} as an F_p-ring is insufficient. Vulnerabilities to physical bit leakage are associated with the binary representation of the field elements. The monic $\deg t$ irreducible polynomial $\Pi(Z)$, such that $F_{p^t} \cong F_p[Z]/\Pi(Z)$, determines this representation. Therefore, the characterization of secure evaluation places must account for this irreducible polynomial $\Pi(Z)$. So, more fundamentally, are there "better" or "worse" choices for the irreducible polynomial?

Investigating our problem for $n > k$ poses non-trivial technical challenges, even for prime fields. For our $n = k$ case, studying leakage-resilience required determining the "similarity" between two square waves. For the $n > k$ case, one needs to determine the "higher-order correlations" among three or more square waves. For example, three square waves can generate eight sign tuples $(\sigma_1, \sigma_2, \sigma_3) \in \{\pm 1\}^3$. Leakage resilience corresponds to the probability of these signs being independent of the secret. Generalizing our technical approach for the $(n, k) = (3, 2)$ case while ensuring $\leqslant 1/\sqrt{p}$ insecurity seems challenging.[1]

Leaking multiple bits from each secret share also encounters similar technical challenges. In particular, characterizing higher-order correlations is required. Furthermore, instead of square waves (where the peaks and troughs have identical length) one needs to analyze more general waves whose peaks and troughs have very different lengths.

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Supporting Materials.

A Solving Simultaneous Diophantine Equations

Figure 5 presents our algorithm. In this section, the "LLL algorithm" refers to the algorithm with the following guarantees.

Imported Theorem 2 (LLL [31, Proposition 1.39]) *There exists a polynomial-time algorithm that, given a positive integer d and rational numbers $r_1, r_2, \ldots, r_d, \varepsilon$ satisfying $0 < \varepsilon < 1$, finds integers $s_1, s_2, \ldots, s_d$, and t for which*

$$|s_i - t \cdot r_i| \leqslant \varepsilon,$$

for $1 \leqslant i \leqslant d$ and $1 \leqslant t \leqslant 2^{d(d+1)/4} \cdot \varepsilon^{-d}$.

Input. $\alpha_1, \alpha_2 \in F^*$, where F is the prime field of order p

Output. Elements $u, v \in F^*$ such that $(u, v) \in [\alpha_1 : \alpha_2]$ and

$$u, v \in \{-B, -(B-1), \ldots, 0, 1, \ldots, (B-1), B\} \mod p,$$

where $B := \left\lceil 2^{3/4} \cdot \sqrt{p} \right\rceil$.

Algorithm.
1. Interpret $\alpha_1, \alpha_2 \in \{0, 1, \ldots, p-1\}$ as positive integers
2. Define $d = 2$
3. Define $r_1 = \alpha_1/p \in \mathbb{Q}$ and $r_2 = \alpha_2/p \in \mathbb{Q}$
4. Define $\varepsilon = B/p \in \mathbb{Q}$
5. Use the LLL algorithm to find integers s_1, s_2, and t
6. Interpret t as an element of F. Define $u = \alpha \cdot t \in F$ and $v = \alpha \cdot t \in F$

Fig. 5. Our Algorithm to obtain (u, v) from (α_1, α_2) using the LLL-algorithm.

Let us proceed to analyze our algorithm of Figure 5. The parameter setting needs to ensure that $t \leqslant 2^{d(d+1)/4}\varepsilon^{-d} < p$. Recall that $\varepsilon = B/p$. Substituting this value and rearranging, one needs to ensure that $2^{d(d+1)/4} \cdot p^{d-1} < B$. Therefore we have chosen $B = \left\lceil 2^{(d+1)/4} p^{1-1/4} \right\rceil$. Consequently, one can interpret t as an F^* element.

By definition, $(u, v) \in [\alpha_1 : \alpha_2]$ because $u = t \cdot \alpha_1$ and $v = t \cdot \alpha_2$. Next, note that

$$|\alpha_1 \cdot t - s_1 \cdot p| \leqslant \varepsilon \cdot p = B, \text{ and } |\alpha_2 \cdot t - s_2 \cdot p| \leqslant \varepsilon \cdot p = B.$$

This argument completes the analysis that for every (α_1, α_2) how we obtain $(u, v) \in [\alpha_1 : \alpha_2]$ such that u and v are "small (positive/negative) numbers."

B Equivalence classes for Evaluation Places

Consider Shamir's secret-sharing scheme among n parties with reconstruction threshold k over the prime field F of order $p \geqslant 3$. The secret-sharing scheme is the Massey secret-sharing scheme [39] corresponding to the (punctured) Reed-Solomon code with evaluation places $(0, \alpha_1, \alpha_2, \ldots, \alpha_n)$. That is, the dealer chooses a random F-polynomial $P(Z)$ of degree $< k$ conditioned on $P(Z = 0)$ being the secret s. Evaluating this polynomial at evaluation places $Z = \alpha_1, \alpha_2, \ldots, \alpha_n$ generates the secret shares $s_1, s_2, \ldots, s_n$, respectively.

Lemma 10 (Equivalence Classes of Evaluation Places). *The (punctured) Reed-Solomon code corresponding to evaluation places* $(0, \alpha_1, \alpha_2, \ldots, \alpha_n)$ *is identical to the (punctured) Reed-Solomon code corresponding to evaluation places* $(0, \Lambda \cdot \alpha_1, \Lambda \cdot \alpha_2, \ldots, \Lambda \cdot \alpha_n)$*, for any* $\Lambda \in F^*$.

This proposition is a consequence of the properties of Generalized Reed-Solomon codes [20, 32]. In particular, since the linear codes are identical, the corresponding Massey secret-sharing schemes have identical resilience/vulnerability to attacks. That is, the $\mathsf{ShamirSS}(n, k, (\alpha_1, \alpha_2, \ldots, \alpha_n))$ and the $\mathsf{ShamirSS}(n, k, (\Lambda \cdot \alpha_1, \Lambda{\cdot}\alpha_2, \ldots, \Lambda{\cdot}\alpha_n))$ secret-sharing schemes have identical resilience/vulnerability to attacks, for any $\Lambda \in F^*$. Therefore, for given distinct evaluation places $\alpha_1, \alpha_2, \ldots, \alpha_n \in F^*$, we define the equivalence class

$$[\alpha_1 \colon \alpha_2 \colon \cdots \colon \alpha_n] := \{(\Lambda \cdot \alpha_1, \Lambda \cdot \alpha_2, \ldots, \Lambda \cdot \alpha_n) : \Lambda \in F^*\}.$$

Determining the security of the evaluation places $(\alpha_1, \ldots, \alpha_n)$ is equivalent to determining the security of *any element* in the equivalence class $[\alpha_1 \colon \cdots \colon \alpha_n]$.

C Proof of Technical Lemmas

C.1 Proof of Lemma 2

Consider the $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ secret-sharing scheme over a prime field F_p. Consider an arbitrary secret $s \in F_p$ and evaluation places $(u, v) \in [\alpha_1 \colon \alpha_2]$.

$$\begin{aligned}
&\mathsf{2SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0)),\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right) \\
&\quad = \sum_{\vec{\ell} \in \{0,1\}^2} \left| \Pr\left[\vec{\mathrm{LSB}}(\mathsf{Share}(0)) = \vec{\ell}\right] - \Pr\left[\vec{\mathrm{LSB}}(\mathsf{Share}(s)) = \vec{\ell}\right] \right| \\
&\quad = \sum_{\vec{\ell} \in \{0,1\}^2} \left| \mathop{\mathbb{E}}_{X}\left[\mathbb{1}_{\mathrm{LSB}^{-1}(\ell_1)}(uX) \cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_2)}(vX)\right] \right. \\
&\qquad\qquad \left. - \mathop{\mathbb{E}}_{X}\left[\mathbb{1}_{\mathrm{LSB}^{-1}(\ell_1)}(uX + s) \cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_2)}(vX + s)\right] \right|
\end{aligned}$$

Claim 1 *For $\ell \in \{0,1\}$ and $X \in F_p$, we have*

$$\mathbb{1}_{\mathrm{LSB}^{-1}(\ell)}(X) = \frac{1}{2}\left(1 + (-1)^{\ell} \cdot \mathrm{sign}_p(X \cdot 2^{-1})\right).$$

Substituting, we get

$$\begin{aligned}
&2\mathsf{SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right) \\
&\quad = \sum_{\vec{\ell} \in \{0,1\}^2} \Bigg| \mathop{\mathbb{E}}_{X}\left[\left(\frac{1 + (-1)^{\ell_1}\,\mathrm{sign}_p(uX \cdot 2^{-1})}{2}\right) \cdot \left(\frac{1 + (-1)^{\ell_2}\,\mathrm{sign}_p(vX \cdot 2^{-1})}{2}\right)\right] \\
&\qquad\quad - \mathop{\mathbb{E}}_{X}\left[\left(\frac{1 + (-1)^{\ell_1}\,\mathrm{sign}_p((uX+s) \cdot 2^{-1})}{2}\right) \cdot \left(\frac{1 + (-1)^{\ell_2}\,\mathrm{sign}_p((vX+s) \cdot 2^{-1})}{2}\right)\right]\Bigg| \\
&\quad = \frac{1}{4} \cdot \sum_{\vec{\ell} \in \{0,1\}^2} \left| \mathop{\mathbb{E}}_{X}\left[\mathrm{sign}_p(uX \cdot 2^{-1}) \cdot \mathrm{sign}_p(vX \cdot 2^{-1})\right] - \mathop{\mathbb{E}}_{X}\left[\mathrm{sign}_p((uX+s) \cdot 2^{-1}) \cdot \mathrm{sign}_p((vX+s) \cdot 2^{-1})\right] \right| \\
&\quad = \left| \mathop{\mathbb{E}}_{X}\left[\mathrm{sign}_p(uX \cdot 2^{-1}) \cdot \mathrm{sign}_p(vX \cdot 2^{-1})\right] - \mathop{\mathbb{E}}_{X}\left[\mathrm{sign}_p((uX+s) \cdot 2^{-1}) \cdot \mathrm{sign}_p((vX+s) \cdot 2^{-1})\right] \right| \\
&\quad = \frac{1}{p} \cdot \left| \sum_{X \in F_p} \mathrm{sign}_p(uX \cdot 2^{-1}) \cdot \mathrm{sign}_p(vX \cdot 2^{-1}) - \sum_{X \in F_p} \mathrm{sign}_p((uX+s) \cdot 2^{-1}) \cdot \mathrm{sign}_p((vX+s) \cdot 2^{-1}) \right| \\
&\quad = \frac{1}{p} \cdot \left| \sum_{Y \in F_p} \mathrm{sign}_p(uY) \cdot \mathrm{sign}_p(vY) - \sum_{Z \in F_p} \mathrm{sign}_p(uZ) \cdot \mathrm{sign}_p(v(Z - s \cdot 2^{-1} \cdot (u^{-1} - v^{-1}))) \right|
\end{aligned}$$

The last step uses the renaming $X \cdot 2^{-1} \mapsto Y$ (an F automorphism) and $(X + s \cdot u^{-1}) \cdot 2^{-1} \mapsto Z$ (an F automorphism).

Therefore,

$$\mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right) = \frac{\left|\varSigma_{u,v}^{(0)} - \varSigma_{u,v}^{(\varDelta)}\right|}{2p},$$

where $\varDelta := \left(s \cdot 2^{-1}\right) \cdot \left(u^{-1} - v^{-1}\right)$, a linear automorphism over F_p.

C.2 Proof of Lemma 3

For $k, \ell, \varDelta \in F_p$, the proof follows directly from our definition of $\varSigma_{k,\ell}^{(\varDelta)}$, $\mathrm{sign}_p(X)$, $\widetilde{\varSigma}_{k,\ell}^{(\varDelta)}$, and $\widetilde{\mathrm{sign}}_p(X)$. The primary observation is that $\mathrm{sign}_p(X) = \widetilde{\mathrm{sign}}_p(X)$, for all

$X \in F_p^*$, and $\widetilde{\text{sign}}_p(X = 0) = 0$.

$$
\begin{aligned}
\Sigma_{k,\ell}^{(\Delta)} &= \sum_{T \in F} \text{sign}_p(kT) \cdot \text{sign}_p(\ell(T - \Delta)) \\
&= \left(\sum_{T \in F} \widetilde{\text{sign}}_p(kT) \cdot \widetilde{\text{sign}}_p(\ell(T - \Delta)) \right) + \left(\sum_{T \in \{0, \Delta\}} \text{sign}_p(kT) \cdot \text{sign}_p(\ell(T - \Delta)) \right) \\
&= \widetilde{\Sigma}_{k,\ell}^{(\Delta)} + \left(\sum_{T \in \{0, \Delta\}} \text{sign}_p(kT) \cdot \text{sign}_p(\ell(T - \Delta)) \right)
\end{aligned}
$$

The expressions above do not require $0, \Delta, \Delta'$ to be distinct.

C.3 Proof of Lemma 4

Claim 2 (Transference Property) *For all $k, \Delta \in F_p, X \in \mathbb{Z}, X = X' \mod p, x = X'/p \in \frac{1}{p} \cdot \mathbb{Z}$, and $\delta = \Delta/p \in \frac{1}{p} \cdot \mathbb{Z}$,*

$$\widetilde{\text{sign}}_p(k \cdot (X - \Delta)) = \varphi(k \cdot (x - \delta)).$$

Claim 3 *For $k, \Delta \in F_p$ and $x \in \frac{1}{p} \cdot \mathbb{Z}$, define $\delta := \frac{\Delta}{p} \in \frac{1}{p} \cdot \mathbb{Z}$ and $\delta' := \frac{\text{sign}_p(\Delta) \cdot |\Delta|_p}{p} \in \frac{1}{p} \cdot \mathbb{Z}$, then*

$$\varphi(k \cdot (x - \delta)) = \varphi(k \cdot (x - \delta')).$$

Proof. Consider the following exhaustive case analysis.

- **Case 1:** $\Delta \in \{0, 1, \ldots, (p-1)/2\}$. In this scenario, $\text{sign}_p(\Delta) = 1, |\Delta|_p = \Delta$ and $\delta = \delta'$. Then, $\varphi(k \cdot (x - \delta)) = \varphi(k \cdot (x - \delta'))$.
- **Case 2:** $\Delta \in \{(p+1)/2, (p+3)/2, \ldots, p-1\}$. In this scenario, $\text{sign}_p(\Delta) = -1, |\Delta|_p = p - \Delta$ and $\delta' = \delta - 1$. Then,

$$
\begin{aligned}
\varphi(k \cdot (x - \delta')) &= \varphi(k \cdot (x - \delta + 1)) \\
&= \text{sign}(\sin(2\pi k \cdot (x - \delta + 1))) \\
&= \text{sign}(\sin(2\pi k \cdot (x - \delta) + 2\pi k)) \\
&= \text{sign}(\sin(2\pi k \cdot (x - \delta))) \\
&= \varphi(k \cdot (x - \delta))
\end{aligned}
$$

□

Claim 4 *For $k \in F_p$ and $x, \delta \in \frac{1}{p} \cdot \mathbb{Z}$, the following holds.*

$$\varphi(k \cdot (x - \delta)) = \text{sign}_p(k) \cdot \varphi(|k|_p \cdot (x - \delta)).$$

Proof. Consider the following exhaustive case analysis.

– **Case 1:** If $k \in \{0, 1, \ldots, (p-1)/2\}$, $|k|_p = k, \mathrm{sign}_p(k) = 1$ and $\varphi(k \cdot (x-\delta)) = \mathrm{sign}_p(k) \cdot \varphi(|k|_p \cdot (x-\delta))$ holds by simply plugging in the values.
– **Case 2:** If $k \in \{(p+1)/2, (p+3)/2, \ldots, p-1\}$, then $|k|_p = p-k$, and $\mathrm{sign}_p(k) = -1$. Substituting in $\mathrm{sign}_p(k) \cdot \varphi(|k|_p \cdot (x-\delta))$, we get

$$\begin{aligned}
\mathrm{sign}_p(k) \cdot \varphi(|k|_p \cdot (x-\delta)) &= \mathrm{sign}(\sin(2\pi |k|_p \cdot (x-\delta))) \qquad (|k|_p = p-k) \\
&= \mathrm{sign}_p(k) \cdot \mathrm{sign}(\sin(2\pi(p-k) \cdot (x-\delta))) \\
&= \mathrm{sign}_p(k) \cdot \mathrm{sign}(\sin(2\pi(px - p\delta) - 2\pi k \cdot (x-\delta))) \\
&\qquad (x, \delta \in \tfrac{1}{p} \cdot \mathbb{Z} \implies px, p\delta \in \mathbb{Z}) \\
&= \mathrm{sign}_p(k) \cdot \mathrm{sign}(\sin(-2\pi k \cdot (x-\delta))) \\
&= -\,\mathrm{sign}_p(k) \cdot \mathrm{sign}(\sin(2\pi k \cdot (x-\delta))) \\
&\qquad (\mathrm{sign}_p(k) = -1) \\
&= \mathrm{sign}(\sin(2\pi k \cdot (x-\delta))) \\
&= \varphi(k \cdot (x-\delta))
\end{aligned}$$

□

Given the Transference Property (Claim 2), Claim 3 and Claim 4, we observe that for $k, \ell, \Delta \in F_p, T \in F, t = T/p \in \mathbb{Q}$ and $\delta = \frac{\mathrm{sign}_p(\Delta) \cdot |\Delta|_p}{p} \in \mathbb{Q}$,

$$\begin{aligned}
\widetilde{\Sigma}^{(\Delta)}_{k,\ell} &= \sum_{T \in F} \widetilde{\mathrm{sign}}_p(kT) \cdot \widetilde{\mathrm{sign}}_p(\ell(T-\Delta)) \\
&= \sum_{t \in \left\{\frac{0}{p}, \frac{1}{p}, \ldots, \frac{p-1}{p}\right\}} \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot \varphi(|k|_p t) \cdot \varphi(|\ell|_p (t-\delta))
\end{aligned}$$

Definition 1 (Number of Oscillations). *A Boolean function $f\colon [0,1] \to \{\pm 1\}$ oscillates at $x \in [0,1)$ if $f(x) \neq \lim_{h \to 0^+} f(x+h)$. The* number of oscillations *is the cardinality of the following set.*

$$\left\{ x\colon f(x) \neq \lim_{h \to 0^+} f(x+h) \right\}.$$

Since our functions are periodic with period 1, counting the number of oscillations in the interval $[0,1)$ in our context suffices.

By straightforward counting, one concludes the following.

Claim 5 (Counting Number of Oscillations) *For any $|k|_p, |\ell|_p \in \{1, \ldots, (p-1)/2\}$,*

1. $\varphi(|\ell|_p (x-\delta))$ *oscillates* $(2|\ell|_p - 1)$ *times, if* $\delta \in \frac{1}{2|\ell|_p} \cdot \mathbb{Z}$
2. $\varphi(|\ell|_p (x-\delta))$ *oscillates* $2|\ell|_p$ *times, if* $\delta \notin \frac{1}{2|\ell|_p} \cdot \mathbb{Z}$
3. $\varphi(|k|_p x) \cdot \varphi(|\ell|_p (x-\delta))$ *oscillates* $2(|k|_p + |\ell|_p) - 2$ *times, if* $\delta \in \frac{1}{2|k|_p} \cdot \mathbb{Z} \cap \frac{1}{2|\ell|_p} \cdot \mathbb{Z}$
4. $\varphi(|k|_p x) \cdot \varphi(|\ell|_p (x-\delta))$ *oscillates* $2(|k|_p + |\ell|_p) - 1$ *times, if* $\delta \notin \frac{1}{2|k|_p} \cdot \mathbb{Z} \cap \frac{1}{2|\ell|_p} \cdot \mathbb{Z}$

We prove a general result connecting Boolean functions' sum and the integral.

Claim 6 (Sum and Integral Connection) *Fix an integer $n \in \{1, 2, \dots\}$. Let $f\colon [0,1] \to \{\pm 1\}$ be a Boolean function that oscillates H times in the range $[0,1]$. Then,*

$$\frac{1}{n} \cdot \sum_{t \in \left\{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n-1}{n}\right\}} f(t) \in \int_0^1 f(t)\,\mathrm{d}t \pm \frac{2H}{n}.$$

Proof. Consider an interval $[r, r+1/n)$, for $r \in \{0/n, 1/n, \dots, (n-1)/n\}$. If f does not oscillate in this interval, then f is constant in the interval, and we conclude

$$\frac{1}{n} \cdot f(t) = \int_r^{r+1/n} f(t)\,\mathrm{d}t.$$

If f oscillates at some point in this interval, then (due to f being Boolean) we conclude

$$\frac{1}{n} \cdot f(t) \in [-1/n, 1/n] \subseteq \int_r^{r+1/n} f(t)\,\mathrm{d}t \pm \frac{2}{n}.$$

Adding over all $r \in \{0/n, 1/n, \dots, (n-1)/n\}$, we get the claim. □

Consider $f(t) = \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot \varphi(|k|_p t) \cdot \varphi(|\ell|_p (t - \delta))$, as a consequence of Claim 5 and Claim 6, we conclude Lemma 3.

For any $k, \ell, \Delta \in F_p$ and $\delta = \frac{\mathrm{sign}_p(\Delta)\cdot|\Delta|_p}{p} \in \mathbb{Q}$.

$$\frac{1}{p} \cdot \widetilde{\Sigma}_{k,\ell}^{(\Delta)} = \begin{cases} \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot I^{(\delta)}_{|k|_p, |\ell|_p} \pm \frac{4(|k|_p + |\ell|_p) - 4}{p} & \text{if } \delta \in \frac{1}{2|k|_p} \cdot \mathbb{Z} \cap \frac{1}{2|\ell|_p} \cdot \mathbb{Z} \\ \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot I^{(\delta)}_{|k|_p, |\ell|_p} \pm \frac{4(|k|_p + |\ell|_p) - 2}{p} & \text{if } \delta \notin \frac{1}{2|k|_p} \cdot \mathbb{Z} \cap \frac{1}{2|\ell|_p} \cdot \mathbb{Z} \end{cases}$$

Combining the two cases, we get

$$\frac{1}{p} \cdot \widetilde{\Sigma}_{k,\ell}^{(\Delta)} = \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot I^{(\delta)}_{|k|_p, |\ell|_p} \pm \frac{4(|k|_p + |\ell|_p) - 2}{p}.$$

C.4 Proof of Lemma 5

To begin, we formalize the orthogonal properties of the sine and cosine functions.

Proposition 3 (Orthogonality of Sine/Cosine Waves [28, Page 38]). *For $k, \ell \in \{1, 2, \dots\}$*

$$\int_0^1 \sin(2k\pi t) \cdot \sin(2\ell\pi t)\,\mathrm{d}t = \begin{cases} 0, & \text{if } k \neq \ell \\ \frac{1}{2}, & \text{if } k = \ell. \end{cases}$$

$$\int_0^1 \sin(2k\pi t) \cdot \cos(2\ell\pi t)\,\mathrm{d}t = 0.$$

For the periodic *square wave* [47, 22, 21] $\varphi\colon \mathbb{R} \to \{-1, 0, +1\}$.

$$\varphi(x) := \operatorname{sign}\sin(2\pi x),$$

[22] uses (basic) Fourier analysis and Proposition 3 to determine the Fourier expansion of $\varphi(x)$.

$$\varphi(x) = \sum_{\text{odd } n>0} \frac{4}{\pi n} \cdot \sin(2n\pi x). \tag{11}$$

We prove the following claim for standardization.

Claim 7 *For $k, \ell \in F_p$ and $\delta \in \mathbb{R}$, the following identity holds*

$$I_{k,\ell}^{(\delta)} = I_{k/g,\ell/g}^{(\delta)},$$

where $g = \gcd(k, \ell)$.

Proof. Define $\psi_{k,\ell}^{(\delta)}(x) := \varphi(kx) \cdot \varphi(\ell \cdot (x - \delta))$.

Observe that $\psi_{k,\ell}^{(\delta)}(x) = \psi_{k,\ell}^{(\delta)}(x + 1/d)$, for any d that divides both k and ℓ. Let $g = \gcd(k, \ell)$. So, from our observation, we conclude that $\psi_{k,\ell}^{(\delta)}$ has period $1/g$. Therefore, we conclude that

$$I_{k,\ell}^{(\delta)} = g \cdot \int_0^{1/g} \psi_{k,\ell}^{(\delta)}(t)\, \mathrm{d}t.$$

Next, note that $\psi_{k,\ell}^{(\delta)}(x) = \psi_{k/d,\ell/d}^{(\delta)}(d \cdot x)$, for any d that divides both k and ℓ. Therefore, we get

$$I_{k,\ell}^{(\delta)} = g \cdot \int_0^{1/g} \psi_{k/g,\ell/g}^{(\delta)}(gt)\, \mathrm{d}t.$$

By substituting the variable $r = gt$, we get

$$I_{k,\ell}^{(\delta)} = g \cdot \int_0^{1} \psi_{k/g,\ell/g}^{(\delta)}(r) \cdot \frac{1}{g} \cdot \mathrm{d}r = I_{k/g,\ell/g}^{(\delta)}.$$

□

Previously only $I_{k,\ell}^{(0)}$ was studied [47, 22]. In particular, motivated by our application scenario, we study $I_{k,\ell}^{(\delta)}$, for all $\delta \in \mathbb{R}$. To begin our analysis, we assume that k and ℓ are relatively prime.

Claim 8 *For relatively prime $k, \ell \in F_p$ such that $k \cdot \ell$ is even, $I_{k,\ell}^{(\delta)} = 0$, for all $\delta \in \mathbb{R}$.*

Proof. Suppose k is even, and ℓ is odd. In this case, for any odd $m, n > 0$, observe that

$$\begin{aligned}
&\sin\left(2n\pi \cdot k\left(\frac{1}{2}+t\right)\right) \cdot \sin\left(2m\pi \cdot \ell\left(\frac{1}{2}+t-\delta\right)\right) \\
&\quad = \sin\begin{pmatrix} 2n\pi \cdot kt \\ + \\ \underbrace{nk}_{\text{even}} \cdot \pi \end{pmatrix} \cdot \sin\begin{pmatrix} 2m\pi \cdot \ell(t-\delta) \\ + \\ \underbrace{m\ell}_{\text{odd}} \cdot \pi \end{pmatrix} \\
&\quad = \sin(2n\pi \cdot kt) \cdot (\ -\sin(2m\pi \cdot \ell(t-\delta))\ )
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^1 \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t &= \int_0^{1/2} \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t \\
&\quad + \int_{1/2}^1 \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t \\
&= \int_0^{1/2} \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t \\
&\quad - \int_0^{1/2} \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t \\
&= 0. \qquad (12)
\end{aligned}$$

Now, we can prove the lemma.

$$\begin{aligned}
I_{k,\ell}^{(\delta)} &= \int_0^1 \varphi(kt) \cdot \varphi(\ell(t-\delta))\,\mathrm{d}t \\
&= \frac{16}{\pi^2} \sum_{\text{odd } n>0} \sum_{\text{odd } m>0} \frac{1}{mn} \int_0^1 \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t \\
&&\text{(By Equation 11)} \\
&= 0 &\text{(By Equation 12)}
\end{aligned}$$

Finally, if k is odd and ℓ is even, then

$$\begin{aligned}
&\sin\left(2n\pi \cdot k\left(\frac{1}{2}+t\right)\right) \cdot \sin\left(2m\pi \cdot \ell\left(\frac{1}{2}+t-\delta\right)\right) \\
&\quad = \sin\begin{pmatrix} 2n\pi \cdot kt \\ + \\ \underbrace{nk}_{\text{odd}} \cdot \pi \end{pmatrix} \cdot \sin\begin{pmatrix} 2m\pi \cdot \ell(t-\delta) \\ + \\ \underbrace{m\ell}_{\text{even}} \cdot \pi \end{pmatrix} \\
&\quad = (\ -\sin(2n\pi \cdot kt)\ ) \cdot \sin(2m\pi \cdot \ell(t-\delta))
\end{aligned}$$

Again, Equation 12 holds, and the proof of this case goes through. □

Claim 9 *For relatively prime $k, \ell \in \{1, 2, \dots\}$ such that $k \cdot \ell$ is odd,*

$$I_{k,\ell}^{(\delta)} = \frac{\cos(2\ell\pi \cdot \delta)}{k\ell},$$

for all $\delta \in \mathbb{R}$. Therefore, $I_{k,\ell}^{(\delta)}$ achieves its maximum at $\delta \in \frac{1}{\ell} \cdot \mathbb{Z}$, and the minimum at $\delta \in \frac{1}{2\ell} + \frac{1}{\ell} \cdot \mathbb{Z}$.

Proof. We begin with a generalization of Proposition 3.

Claim.

$$\int_0^1 \sin(2k\pi t) \cdot \sin(2\ell\pi(t - \delta))\, \mathrm{d}t = \begin{cases} 0, & \text{if } k \neq \ell \\ \frac{1}{2}\cos(2\ell\pi\delta), & \text{if } k = \ell. \end{cases}$$

Proof (of the claim above).

$$\begin{aligned} \int_0^1 \sin(2k\pi t) \cdot \sin(2\ell\pi(t - \delta))\, \mathrm{d}t &= \int_0^1 \sin(2k\pi t) \cdot \sin(2\ell\pi t)\cos(2\ell\pi\delta)\, \mathrm{d}t \\ &\quad - \int_0^1 \sin(2k\pi t) \cdot \cos(2\ell\pi t)\sin(2\ell\pi\delta)\, \mathrm{d}t \\ &= \cos(2\ell\pi\delta) \int_0^1 \sin(2k\pi t) \cdot \sin(2\ell\pi t)\, \mathrm{d}t, \end{aligned}$$

because, for all $k, \ell \in \{1, 2, \dots\}$, Proposition 3 implies

$$\int_0^1 \sin(2k\pi t) \cdot \cos(2\ell\pi t)\, \mathrm{d}t = 0.$$

The proof of our claim follows from Proposition 3 because $\int_0^1 \sin(2k\pi t) \cdot \sin(2\ell\pi t)\, \mathrm{d}t = 1/2$ if (and only if) $k = \ell$; otherwise, it is 0. □

Next, we simplify the expression for $I_{k,\ell}^{(\delta)}$.

$$\begin{aligned} I_{k,\ell}^{(\delta)} &= \int_0^1 \varphi(kt) \cdot \varphi(\ell(t - \delta))\, \mathrm{d}t \\ &= \frac{16}{\pi^2} \sum_{\text{odd } n>0} \sum_{\text{odd } m>0} \frac{1}{mn} \int_0^1 \sin(2n\pi \cdot kt)\sin(2m\pi \cdot \ell(t - \delta))\, \mathrm{d}t \end{aligned}$$

(By Equation 11)

In light of the claim above, the integral in the RHS survives if and only if $nk = m\ell$. Since, $\gcd(k, \ell) = 1$, note that $nk = m\ell$ if and only if

$$(n, m) \in J := \Big\{ (\ell, k), (3\ell, 3k), (5\ell, 5k), \dots \Big\}.$$

With this observation and Proposition 3, we get

$$\begin{aligned}
I_{k,\ell}^{(\delta)} &= \frac{16}{\pi^2} \sum_{(n,m)\in J} \frac{\cos(2\ell\pi\delta)}{mn} \int_0^1 \sin(2n\pi \cdot kt)\sin(2m\pi \cdot \ell t)\,\mathrm{d}t \\
&= \frac{16}{\pi^2} \sum_{\text{odd } a>0} \frac{\cos(2\ell\pi\delta)}{k\ell \cdot a^2} \int_0^1 \sin(2k\ell a\pi \cdot t)\sin(2k\ell a\pi \cdot t)\,\mathrm{d}t \\
&= \frac{16}{\pi^2} \cdot \frac{1}{k\ell} \sum_{\text{odd } a>0} \frac{1}{a^2} \cdot \frac{\cos(2\ell\pi\delta)}{2} && \text{(By Proposition 3)} \\
&= \frac{\cos(2\ell\pi\delta)}{k\ell} \cdot \frac{\not{8}}{\not{\pi}^2} \cdot \frac{\pi^{\not{2}}}{\not{8}} = \frac{\cos(2\ell\pi\delta)}{k\ell} && \left(\text{Because} \sum_{\text{odd } a>0} \tfrac{1}{a^2} = \tfrac{3}{4}\cdot\zeta(2) = \tfrac{\pi^2}{8}\right)
\end{aligned}$$

Combining Claim 8 and Claim 9, we showed that for relatively prime $k, \ell \in F_p$,

$$I_{k,\ell}^{(\delta)} = \begin{cases} 0 & \text{if } k\cdot\ell \text{ is even} \\ \frac{\cos(2\ell\pi\cdot\delta)}{k\ell} & \text{if } k\cdot\ell \text{ is odd} \end{cases}.$$

Claim 7 generalizes the result to all $k, \ell \in F_p$ by considering $g = \gcd(k, \ell)$. This proves our lemma Lemma 5 that

$$I_{k,\ell}^{(\delta)} = \begin{cases} 0 & \text{if } k\cdot\ell \text{ is even} \\ \frac{g^2}{k\ell}\cdot\cos(2\ell\pi\cdot\delta) & \text{if } k\cdot\ell \text{ is odd} \end{cases}.$$

□

C.5 Proof of Lemma 1

Consider evaluation places $(u, v) \in [\alpha_1 \colon \alpha_2]$ and secret $s \in F_p$.

Define $\Delta := \left(s \cdot 2^{-1}\right) \cdot \left(u^{-1} - v^{-1}\right) \in F_p$.

Lemma 2 shows that

$$\mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right) = \frac{\left|\Sigma_{u,v}^{(0)} - \Sigma_{u,v}^{(\Delta)}\right|}{2p} \tag{13}$$

Lemma 3 proves

$$\Sigma_{u,v}^{(\Delta)} = \widetilde{\Sigma}_{u,v}^{(\Delta)} + \left(\sum_{T\in\{0,\Delta\}} \mathrm{sign}_p(uT)\cdot\mathrm{sign}_p(v(T-\Delta))\right). \tag{14}$$

Apply Lemma 3 to Equation 13,

$$\begin{aligned}
\mathsf{SD}\left(\mathsf{LSB}(\mathsf{Share}(0))\ ,\ \mathsf{LSB}(\mathsf{Share}(s))\right) &= \frac{\left|\widetilde{\Sigma}^{(0)}_{u,v} - \widetilde{\Sigma}^{(\Delta)}_{u,v} - \left(\sum_{T\in\{0,\Delta\}\setminus\{0\}} \mathrm{sign}_p(uT)\cdot \mathrm{sign}_p(v(T-\Delta))\right)\right|}{2p} \\
&= \frac{\left|\widetilde{\Sigma}^{(0)}_{u,v} - \widetilde{\Sigma}^{(\Delta)}_{u,v} - \mathrm{sign}_p(u\Delta)\right|}{2p} \\
&= \frac{\left|\widetilde{\Sigma}^{(0)}_{u,v} - \widetilde{\Sigma}^{(\Delta)}_{u,v}\right|}{2p} \pm \frac{1}{2p} \qquad (15)
\end{aligned}$$

Define $\delta = \frac{\mathrm{sign}_p(\Delta)\cdot|\Delta|_p}{p} \in \mathbb{Q}$.
Lemma 4 states that

$$\frac{1}{p}\cdot \widetilde{\Sigma}^{(\Delta)}_{u,v} = \mathrm{sign}_p(u)\cdot \mathrm{sign}_p(v)\cdot I^{(\delta)}_{|u|_p,|v|_p} \pm \frac{4(|u|_p+|v|_p)-2}{p} \qquad (16)$$

.

Apply Lemma 4 to Equation 15,

$$\mathsf{SD}\left(\mathsf{LSB}(\mathsf{Share}(0))\ ,\ \mathsf{LSB}(\mathsf{Share}(s))\right) = \frac{\left|I^{(0)}_{|u|_p,|v|_p} - I^{(\delta)}_{|u|_p,|v|_p}\right|}{2} \pm \frac{4(|u|_p+|v|_p)-2}{p} \pm \frac{1}{2p}. \qquad (17)$$

Lemma 5 proves that for $g = \gcd(|u|_p, |v|_p)$,

$$I^{(\delta)}_{|u|_p,|v|_p} = \begin{cases} 0, & \text{if } |u|_p\cdot|v|_p/g^2 \text{ is even} \\ \frac{g^2}{|u|_p\cdot|v|_p}\cdot\cos(2|v|_p\pi\cdot\delta), & \text{if } |u|_p\cdot|v|_p/g^2 \text{ is odd.} \end{cases},$$

Finally, apply Lemma 5 to Equation 17.

$$\mathsf{SD}\left(\mathsf{LSB}(\mathsf{Share}(0))\ ,\ \mathsf{LSB}(\mathsf{Share}(s))\right) =$$
$$\begin{cases} \pm\frac{4(|u|_p+|v|_p)-(3/2)}{p}, & \text{if } |u|_p\cdot|v|_p/g^2 \text{ is even} \\ (1-\cos(2|v|_p\pi\cdot\delta))\cdot\frac{g^2}{2\cdot|u|_p\cdot|v|_p} \pm \frac{4(|u|_p+|v|_p)-(3/2)}{p}, & \text{if } |u|_p\cdot|v|_p/g^2 \text{ is odd,} \end{cases}$$

Replace $(1-\cos(2|v|_p\pi\cdot\delta))$ with $2\sin^2(|v|_p\pi\cdot\delta)$ concludes our proof.

C.6 Proof of Theorem 1

Consider the $\mathsf{ShamirSS}(2,2,(\alpha_1,\alpha_2))$ secret-sharing scheme over a prime field F_p. For $(u,v)\in[\alpha_1,\alpha_2]$, $g=\gcd(|u|_p,|v|_p)$, $\Delta=(s\cdot 2^{-1})\cdot(u^{-1}-v^{-1})\in F_p^*$ and

$\delta = \frac{\mathrm{sign}_p(\varDelta)\cdot|\varDelta|_p}{p} \in \mathbb{Q}$,

$$\mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right) =$$

$$\begin{cases} \pm\frac{4(|u|_p+|v|_p)-(3/2)}{p}, & \text{if } |u|_p\cdot|v|_p/g^2 \text{ is even} \\ (1-\cos(2|v|_p\pi\cdot\delta))\cdot\frac{g^2}{2\cdot|u|_p\cdot|v|_p} \pm \frac{4(|u|_p+|v|_p)-(3/2)}{p}, & \text{if } |u|_p\cdot|v|_p/g^2 \text{ is odd.} \end{cases}$$

If $|u|_p \cdot |v|_p/g^2$ is even, then

$$\max_{s\in F} \mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right) = \pm\frac{4(|u|_p+|v|_p)-(3/2)}{p}.$$

If $|u|_p \cdot |v|_p/g^2$ is odd, then $\max_{s\in F} \mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right)$ is achieved when $\cos(2|v|_p\pi\cdot\delta)$ is closest to -1.

Claim 10 *For prime $p \geqslant 3$ and $v, \varDelta \in F_p$,*

$$\cos\left(2\pi\cdot\frac{\mathrm{sign}_p(\varDelta)|\varDelta|_p|v|_p}{p}\right) = \cos\left(2\pi\cdot\frac{(\varDelta\cdot v) \mod p}{p}\right).$$

By Claim 10, $\cos(2|v|_p\pi\cdot\delta) = \cos\left(2\pi\cdot\frac{(\varDelta\cdot v) \mod p}{p}\right)$.

For $\varDelta = (s\cdot 2^{-1})\cdot(u^{-1}-v^{-1}) \in F_p^*$ and $v \in F_p$, $\varDelta\cdot v = (s\cdot 2^{-1})\cdot(u^{-1}v-1) \in F_p$.

$\min_{s\in F_p^*}\cos(2|v|_p\pi\cdot\delta)$ is achieved when $\varDelta\cdot v = (s\cdot 2^{-1})\cdot(u^{-1}v-1) = (p-1)/2 \in F_p$ or $\varDelta\cdot v = (s\cdot 2^{-1})\cdot(u^{-1}v-1) = (p+1)/2 \in F_p$ which is equivalent as $s = \pm(u^{-1}v-1)^{-1} \in F_p^*$.

When $\varDelta\cdot v = (p-1)/2 \in F_p$,

$$\cos(2|v|_p\pi\cdot\delta) = \cos\left(2\pi\frac{(p-1)/2}{p}\right) = -\cos(\pi/p).$$

Similarly, when $\varDelta\cdot v = (p+1)/2 \in F_p$,

$$\cos(2|v|_p\pi\cdot\delta) = \cos\left(2\pi\frac{(p+1)/2}{p}\right) = -\cos(\pi/p).$$

Therefore, if $|u|_p\cdot|v|_p/g^2$ is odd, then

$$\max_{s\in F} \mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right) = \left(\frac{1+\cos(\pi/p)}{2}\right)\cdot\frac{g^2}{|u|_p\cdot|v|_p} \quad \pm\frac{4(|u|_p+|v|_p)-(3/2)}{p}$$

and the maximum is achieved when $s = \pm(u^{-1}v-1)^{-1} \in F_p^*$.

Efficient distinguished construction. Consider the following security game (illustrated in the figure below). The attacker picks a secret $s \in F_p^*$ and sends it to the challenger. The challenger picks a random bit $b \in \{0,1\}$. If $b = 0$, the challenger samples (ℓ_1, ℓ_2) from distribution $D_0 := \vec{\mathrm{LSB}}(\mathsf{Share}(0))$ and sends it to the attacker. Otherwise, the challenger samples (ℓ_1, ℓ_2) from distribution $D_1 := \mathrm{L\vec{S}B}(\mathsf{Share}(s))$ and sends it to the attacker. The attacker aims to guess which distribution (ℓ_1, ℓ_2) is sampled from. It uses the maximum likelihood decoder and then returns its guess $\tilde{b}$ to the challenger. The attacker wins the security game if $b = \tilde{b}$.

Attacker		**Challenger**
	$\xrightarrow{s^* \in F_p^*}$	$D_0 = \mathrm{L\vec{S}B}(\mathsf{Share}(0))$
		$D_1 = \mathrm{L\vec{S}B}(\mathsf{Share}(s^*))$
		$b \leftarrow\$ \{0, 1\}$
$\tilde{b} = \mathrm{ML}(\ell_1, \ell_2)$	$\xleftarrow{(\ell_1, \ell_2)}$	$(\ell_1, \ell_2) \leftarrow\$ D_b$
	$\xrightarrow{\tilde{b}}$	$b == \tilde{b}$

The maximum likelihood distinguisher outputs $\tilde{b} = 0$ if $\Pr[(\ell_1, \ell_2)|s = 0] \geqslant \Pr[(\ell_1, \ell_2)|s = s^*]$ and $\tilde{b} = 1$ if $\Pr[(\ell_1, \ell_2)|s = 0] < \Pr[(\ell_1, \ell_2)|s = s^*]$. The output depends on $\mathrm{sign}\left(\Pr[(\ell_1, \ell_2)|s = 0] - \Pr[(\ell_1, \ell_2)|s = s^*]\right)$.

For evaluation places (u, v), where $|u|_p \cdot |v|_p / g^2$ is odd and $g = \gcd(|u|_p, |v|_p)$, and $\varDelta = (s^* \cdot 2^{-1}) \cdot (u^{-1} - v^{-1}) \in F^*$, we get

$$
\begin{aligned}
&\Pr[(\ell_1, \ell_2)|s = 0] - \Pr[(\ell_1, \ell_2)|s = s^*] \\
&= (-1)^{\ell_1 + \ell_2} \cdot \frac{\varSigma_{u,v}^{(0)} - \varSigma_{u,v}^{(\varDelta)}}{4p} && \text{(Supporting Material C.1)} \\
&= (-1)^{\ell_1 + \ell_2} \cdot \frac{\widetilde{\varSigma}_{u,v}^{(0)} - \widetilde{\varSigma}_{u,v}^{(\varDelta)} - \mathrm{sign}_p(u\varDelta)}{4p} && \text{(Lemma 3)} \\
&= (-1)^{\ell_1 + \ell_2} \cdot \frac{\widetilde{\varSigma}_{u,v}^{(0)} - \widetilde{\varSigma}_{u,v}^{(\varDelta)}}{4p} \pm \frac{1}{4p} \\
&= \frac{(-1)^{\ell_1 + \ell_2} \cdot \mathrm{sign}_p(u) \cdot \mathrm{sign}_p(v)}{4} \cdot \left(I_{|u|_p, |v|_p}^{(0)} - I_{|u|_p, |v|_p}^{(\delta)} \pm 2 \cdot \frac{4(|u|_p + |v|_p) - (3/2)}{p} \right) \\
& && \text{(Lemma 4, } \delta = \tfrac{\mathrm{sign}_p(\varDelta)|\varDelta|_p}{p}\text{)} \\
&= \frac{(-1)^{\ell_1 + \ell_2} \cdot \mathrm{sign}_p(u) \cdot \mathrm{sign}_p(v)}{4} \cdot \left(\sin^2(|v|_p \pi \cdot \delta) \cdot \frac{g^2}{|u|_p \cdot |v|_p} \pm 2 \cdot \frac{4(|u|_p + |v|_p) - (3/2)}{p} \right) \\
& && \text{(Lemma 5)}
\end{aligned}
$$

Consider attacker picks $s = \pm(u^{-1} \cdot v - 1)^{-1} \in F^*$ such that

$$
\begin{aligned}
&\Pr[(\ell_1, \ell_2)|s = 0] - \Pr[(\ell_1, \ell_2)|s = s^*] \\
=&\frac{(-1)^{\ell_1 + \ell_2} \cdot \mathrm{sign}_p(u) \cdot \mathrm{sign}_p(v)}{4} \cdot \left(\cos^2(\pi/2p) \cdot \frac{g^2}{|u|_p \cdot |v|_p} \pm 2 \cdot \frac{4(|u|_p + |v|_p) - (3/2)}{p} \right)
\end{aligned}
$$

Since $\mathsf{SD}\left(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\right) > \frac{4(|u|_p+|v|_p)-(3/2)}{p}$ by our assumption, then

$$\cos^2(\pi/2p)\cdot\frac{g^2}{|u|_p\cdot|v|_p} - 2\cdot\frac{4(|u|_p+|v|_p)-(3/2)}{p} > 0.$$

Hence,

$$\mathrm{sign}\left(\Pr[(\ell_1,\ell_2)|s=0] - \Pr[(\ell_1,\ell_2)|s=s^*]\right) = (-1)^{\ell_1+\ell_2}\cdot\mathrm{sign}_p(u)\cdot\mathrm{sign}_p(v).$$

There exists an efficient maximum likelihood distinguisher computing $(-1)^{\ell_1+\ell_2}\cdot\mathrm{sign}_p(u)\cdot\mathrm{sign}_p(v)$. If $(-1)^{\ell_1+\ell_2}\cdot\mathrm{sign}_p(u)\cdot\mathrm{sign}_p(v)>0$, then the maximum likelihood distinguisher outputs $\tilde{b}=0$. Otherwise, it outputs $\tilde{b}=1$.

C.7 Proof of inequality in Corollary 2

Our objective is to prove the following inequality for primes $p \geqslant 11$.

$$\frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p+\frac{3}{2}\cdot\sqrt{p}+\frac{1}{2}}{p-2} \leqslant \frac{\ln p}{4\sqrt{p}}+\frac{5/2}{\sqrt{p}}.$$

We simplify this inequality into a simpler equivalent inequality.

$$\begin{aligned}&\frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p+\frac{3}{2}\cdot\sqrt{p}+\frac{1}{2}}{p-2} \leqslant \frac{\ln p}{4\sqrt{p}}+\frac{5/2}{\sqrt{p}}\\ \iff\ & \cancel{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p}+\cancel{\frac{3}{2}}\cdot\sqrt{p}+\frac{1}{2} \leqslant \cancel{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p}+\cancelto{1}{\frac{5}{2}}\cdot\sqrt{p}-\frac{1}{2}\cdot\frac{\ln p}{\sqrt{p}}-\frac{5}{\sqrt{p}}\\ \iff\ & \frac{1}{2}\sqrt{p}+\frac{1}{2}\ln p \leqslant \sqrt{p} \leqslant p-5.\end{aligned}$$

Thus, it suffices to prove the final inequality. Toward this objective, observe that

1. $\ln p \leqslant \sqrt{p}$, for $p \geqslant 2$, and
2. $\sqrt{p} \leqslant p-5$, for $p \geqslant 11$.

Then, for $p \geqslant 11$,

$$\frac{1}{2}\sqrt{p}+\frac{1}{2}\ln p \leqslant \sqrt{p} \leqslant p-5.$$

Therefore,

$$\frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p+\frac{3}{2}\cdot\sqrt{p}+\frac{1}{2}}{p-2} \leqslant \frac{\ln p}{4\sqrt{p}}+\frac{5/2}{\sqrt{p}}$$

for all $p \geqslant 11$.

C.8 Additional Proof for Corollary 3

Claim 11 *For* $\mathsf{ShamirSS}(2, 2, \vec{\alpha} = (\alpha_1, \alpha_2))$ *and secret* $s \in F$*, define* $\mathrm{err} := \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}$*. Consider* $|\alpha_1|_p, |\alpha_2|_p < \lceil 8^{1/4}\sqrt{p} \rceil$ *and* $|\alpha_1|_p \cdot |\alpha_2|_p / g^2$ *is odd with* $g = \gcd(|\alpha_1|_p, |\alpha_2|_p)$*. When* $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) > 2 \cdot \mathrm{err}$,

$$\mathrm{sign}\left(\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha})\right) = \mathrm{sign}\left(\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}\right)$$

where $\Delta := (s \cdot 2^{-1}) \cdot (\alpha_1^{-1} - \alpha_2^{-1}) \in F$.

Proof.

$$\begin{aligned}
\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p} &= \frac{\widetilde{\Sigma}^{(0)}_{\alpha_1,\alpha_2} - \widetilde{\Sigma}^{(\Delta)}_{\alpha_1,\alpha_2} - \mathrm{sign}_p(\alpha_1 \Delta)}{p} && \text{(Lemma 3)} \\
&= \mathrm{sign}_p(\alpha_1) \cdot \mathrm{sign}_p(\alpha_2) \cdot \left(I^{(0)}_{|\alpha_1|_p,|\alpha_2|_p} - I^{(\delta)}_{|\alpha_1|_p,|\alpha_2|_p}\right) \pm 2 \cdot \frac{4(|\alpha_1|_p + |\alpha_2|_p) - (3/2)}{p} \\
& && \left(\text{Lemma 4, } \delta = \frac{\mathrm{sign}_p(\Delta)|\Delta|_p}{p}\right)
\end{aligned}$$

Equivalently,

$$\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p} \pm 2 \cdot \frac{4(|\alpha_1|_p + |\alpha_2|_p) - (3/2)}{p} = \mathrm{sign}_p(\alpha_1) \cdot \mathrm{sign}_p(\alpha_2) \cdot \left(I^{(0)}_{|\alpha_1|_p,|\alpha_2|_p} - I^{(\delta)}_{|\alpha_1|_p,|\alpha_2|_p}\right).$$

For $|\alpha_1|_p, |\alpha_2|_p < \lceil 8^{1/4}\sqrt{p} \rceil$,

$$2 \cdot \frac{4(|\alpha_1|_p + |\alpha_2|_p) - (3/2)}{p} \leqslant 2 \cdot \mathrm{err} < \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) = \frac{\left|\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}\right|}{p}.$$

which implies that $\pm 2 \cdot \frac{4(|\alpha_1|_p + |\alpha_2|_p) - (3/2)}{p}$ does not change the sign of $\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}$,

$$\mathrm{sign}\left(\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p} \pm 2 \cdot \frac{4(|\alpha_1|_p + |\alpha_2|_p) - (3/2)}{p}\right) = \mathrm{sign}\left(\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}\right)$$

Hence,

$$\begin{aligned}
\mathrm{sign}\left(\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}\right) &= \mathrm{sign}\left(\mathrm{sign}_p(\alpha_1) \cdot \mathrm{sign}_p(\alpha_2) \cdot \left(I^{(0)}_{|\alpha_1|_p,|\alpha_2|_p} - I^{(\delta)}_{|\alpha_1|_p,|\alpha_2|_p}\right)\right) \\
&= \mathrm{sign}\left(\mathrm{sign}_p(\alpha_1) \cdot \mathrm{sign}_p(\alpha_2) \cdot \left(\sin^2(|v|_p \pi \cdot \delta) \cdot \frac{g^2}{|u|_p \cdot |v|_p}\right)\right) \\
& && \text{(Lemma 5)} \\
&= \mathrm{sign}\left(\mathrm{sign}_p(\alpha_1) \cdot \mathrm{sign}_p(\alpha_2)\right) \\
& && \left(\sin^2(|v|_p \pi \cdot \delta) \cdot \tfrac{g^2}{|u|_p \cdot |v|_p} > 0\right) \\
&= \mathrm{sign}\left(\varepsilon_{\mathrm{LSB}}^{(\mathrm{OUR})}(\vec{\alpha})\right)
\end{aligned}$$

.

□

For any secret $s \in F$, let us first define the distinguishing advantage of the maximum likelihood decoder as

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}; s) := \frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}$$

where $\Delta := (s \cdot 2^{-1}) \cdot (\alpha_1^{-1} - \alpha_2^{-1}) \in F$ and the estimate $\varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s) \in [0,1]$ satisfying

$$\varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s) = \varepsilon_{\mathrm{LSB}}(\vec{\alpha}; s) \quad \pm \mathrm{err}$$

where $\mathrm{err} := \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}$. Given Claim 11, we know that for any secret $s \in F$,

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}; s) \geqslant \varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s) - \mathrm{err}. \tag{18}$$

and

$$\varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s) \geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}; s) - \mathrm{err}. \tag{19}$$

Consider secret $s^* \in F$ that achieves the maximum $\varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s)$, we define $\varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s^*)$ as follows

$$\varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}) := \max_{s \in F} \varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s) = \varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s^*).$$

Similarly, consider $\tilde{s}^* \in F$ that reaches maximum $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}; s)$, we define $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}; s^*)$ as

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) := \max_{s \in F} \varepsilon_{\mathrm{LSB}}(\vec{\alpha}; s) = \varepsilon_{\mathrm{LSB}}(\vec{\alpha}; \tilde{s}^*).$$

$$\begin{aligned}
\varepsilon_{\mathrm{LSB}}(\vec{\alpha}; s^*) \geqslant & \varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s^*) - \mathrm{err} = \varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}) - \mathrm{err} && \text{(Equation 18)} \\
\geqslant & \varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; \tilde{s}^*) - \mathrm{err} \qquad (\varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s^*) = \max\nolimits_{s \in F} \varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}; s)) && \\
\geqslant & \varepsilon_{\mathrm{LSB}}(\vec{\alpha}; \tilde{s}^*) - 2 \cdot \mathrm{err} && \text{(Equation 19)} \\
= & \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) - 2 \cdot \mathrm{err} > 0 &&
\end{aligned}$$

Therefore, the distinguishing advantage of the maximum likelihood decoder is

$$\geqslant \varepsilon^{(\mathrm{OUR})}_{\mathrm{LSB}}(\vec{\alpha}) - \mathrm{err} \geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) - 2 \cdot \mathrm{err}.$$

C.9 Proof of Lemma 7

$$
\begin{aligned}
&\mathsf{2SD}\left(\vec{\mathrm{LSB}}_{i,j}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}_{i,j}(\mathsf{Share}(s))\right)\\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^2} \left|\Pr\left[\vec{\mathrm{LSB}}_{i,j}(\mathsf{Share}(0)) = \vec{\ell}\right] - \Pr\left[\vec{\mathrm{LSB}}_{i,j}(\mathsf{Share}(s)) = \vec{\ell}\right]\right|\\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^2} \left|\mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{\mathrm{LSB}_i^{-1}(\ell_1)}(\alpha_1 x)\cdot \mathbb{1}_{\mathrm{LSB}_j^{-1}(\ell_2)}(\alpha_2 x)\right]\right.\\
&\qquad\qquad\qquad \left. - \mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{\mathrm{LSB}_i^{-1}(\ell_1)}(\alpha_1 x + s)\cdot \mathbb{1}_{\mathrm{LSB}_j^{-1}(\ell_2)}(\alpha_2 x + s)\right]\right|\\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^2} \left|\mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{\mathrm{LSB}_i^{-1}(0)}(\alpha_1 x)\cdot \mathbb{1}_{\mathrm{LSB}_j^{-1}(0)}(\alpha_2 x)\right]\right.\\
&\qquad\qquad\qquad \left. - \mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{\mathrm{LSB}_i^{-1}(0)}(\alpha_1 x + s)\cdot \mathbb{1}_{\mathrm{LSB}_j^{-1}(0)}(\alpha_2 x + s)\right]\right|\\
&\qquad\qquad\qquad\qquad\qquad (\text{Using the fact that } \mathbb{1}_{\mathrm{LSB}_k^{-1}(1)} = 1 - \mathbb{1}_{\mathrm{LSB}_k^{-1}(0)})\\
&\quad = 4\cdot\left|\mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{E_i}(\alpha_1 x)\cdot \mathbb{1}_{E_j}(\alpha_2 x)\right] - \mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{E_i}(\alpha_1 x + s)\cdot \mathbb{1}_{E_j}(\alpha_2 x + s)\right]\right|\\
&\quad = 4\cdot\left|\mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{E\cdot(2^i)}(\alpha_1 x)\cdot \mathbb{1}_{E\cdot(2^j)}(\alpha_2 x)\right] - \mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{E\cdot(2^i)}(\alpha_1 x + s)\cdot \mathbb{1}_{E\cdot(2^j)}(\alpha_2 x + s)\right]\right|\\
&\qquad\qquad\qquad\qquad\qquad (\text{By Proposition 2})\\
&\quad = 4\cdot\left|\mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{E}(2^{-i}\alpha_1 x)\cdot \mathbb{1}_{E}(2^{-j}\alpha_2 x)\right]\right.\\
&\qquad\qquad\qquad \left. - \mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{E}(2^{-i}\alpha_1 x + 2^{-i}s)\cdot \mathbb{1}_{E}(2^{-j}\alpha_2 x + 2^{-j}s)\right]\right| \qquad (20)
\end{aligned}
$$

At this point, we introduce the following variable renaming.

Claim 12 *The quantity*

$$\mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{E}(2^{-i}\alpha_1 x + 2^{-i}s)\cdot \mathbb{1}_{E}(2^{-j}\alpha_2 x + 2^{-j}s)\right]$$

is identical to

$$\mathop{\mathbb{E}}_{y}\left[\mathbb{1}_{E}(2^{-i}\alpha_1 y + s')\cdot \mathbb{1}_{E}(2^{-j}\alpha_2 y + s')\right],$$

where

$$y := x + \frac{2^{-i} - 2^{-j}}{2^{-i}\alpha_1 - 2^{-j}\alpha_2}, \qquad \textit{and} \qquad s' := \frac{2^{-i}2^{-j}(\alpha_1 - \alpha_2)}{2^{-i}\alpha_1 - 2^{-j}\alpha_2}\cdot s$$

The proof of this claim is by direct substitution. Note that $s \mapsto s'$ is an automorphism over F^*. We continue the derivation from the expression in Equation 23 as follows.

$$
\begin{aligned}
&= 4 \cdot \Big| \mathop{\mathbb{E}}_{x} \left[\mathbb{1}_E(2^{-i}\alpha_1 x) \cdot \mathbb{1}_E(2^{-j}\alpha_2 x) \right] \\
&\qquad\qquad - \mathop{\mathbb{E}}_{y} \left[\mathbb{1}_E(2^{-i}\alpha_1 y + s') \cdot \mathbb{1}_E(2^{-j}\alpha_2 y + s') \right] \Big| \\
&= \varepsilon_{\mathrm{LSB}}(2^{-i}\alpha_1, 2^{-j}\alpha_2).
\end{aligned}
$$

Therefore, we conclude that the insecurity of $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ secret-sharing scheme against the $\vec{\mathrm{LSB}}_{i,j}$ is identical to the insecurity of the $\mathsf{ShamirSS}(2, 2, (2^{-i}\alpha_1, 2^{-j}\alpha_2))$ secret-sharing scheme against the LSB attack.

C.10 Proof of maximum insecurity bound in Corollary 7

Observe that $\lambda - \lfloor \lambda/2 \rfloor = \lceil \lambda/2 \rceil \geqslant \lfloor \lambda/2 \rfloor$. Therefore, for $1 \leqslant i \leqslant \lambda - 1$, we have

$$
\varepsilon_{\mathrm{LSB}}(2^i \cdot \alpha_1, \alpha_2) \leqslant \frac{4 \cdot 2^t + 4 \cdot (2^{\lambda - t} - 1) - 2}{p}.
$$

All that remains is to prove that this upper bound also holds for $\varepsilon_{\mathrm{LSB}}(2^0 \cdot \alpha_1, \alpha_2)$.

For $\lambda = 2$, we have $t = 1$. In this case, one can verify that the upper bound holds.

$$
\varepsilon(2^0 \cdot \alpha_1, \alpha_2) \leqslant \frac{4 \cdot 2^t + 4 \cdot (2^{\lambda - t} - 1) - 2}{p}.
$$

For $\lambda \geqslant 3$, note that if p is a Mersenne prime, then λ must be odd. Therefore, we have $\lambda - t = t + 1$ and $p = 2^{2t+1} - 1$. Therefore, we need to prove that

$$
\varepsilon(2^0 \cdot \alpha_1, \alpha_2) = \frac{1}{2^t - 1} + \frac{4 + 4 \cdot (2^t - 1) - 2}{p} \leqslant \frac{4 \cdot 2^t + 4 \cdot (2^{t+1} - 1) - 2}{p}.
$$

This bound is equivalent to proving

$$
\begin{aligned}
& & \frac{1}{2^t - 1} &\leqslant \frac{4 \cdot (2^{t+1} - 1)}{2^{2t+1} - 1} & \\
&\iff & \frac{1}{T - 1} &\leqslant \frac{4 \cdot (2T - 1)}{2T^2 - 1} & \text{(substitute } T = 2^t\text{)} \\
&\iff & 0 &\leqslant 6T^2 - 12T + 5 & \\
&\iff & 1/6 &\leqslant (T - 1)^2, &
\end{aligned}
$$

which is true for all $t \geqslant 1$.

So, the overall maximum is

$$
\frac{4 \cdot 2^{\lfloor \lambda/2 \rfloor} + 4 \cdot (2^{\lceil \lambda/2 \rceil} - 1) - 2}{p} = \frac{4 \cdot (2^{\lfloor \lambda/2 \rfloor} + 2^{\lceil \lambda/2 \rceil}) - 6}{p}.
$$

D Proof of Theorem 2

D.1 Fourier Basics

Fourier Basics We use Fourier analysis on prime field F of order p. Define $\omega := \exp(2\pi\imath/p)$. For any functions $f, g\colon F \to \mathbb{C}$, we define the inner product as

$$\langle f, g \rangle := \frac{1}{p} \sum_{x \in F} f(x) \cdot \overline{g(x)},$$

where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. For $z \in \mathbb{C}$, $|z| := \sqrt{z\overline{z}}$. For any $\alpha \in F$, define the function $\widehat{f}\colon F \to \mathbb{C}$ as follows.

$$\widehat{f}(\alpha) := \frac{1}{p} \sum_{x \in F} f(x) \cdot \omega^{-\alpha x}.$$

The Fourier transform maps the function f to the function $\widehat{f}$.

Lemma 11 (Fourier Inversion Formula). $f(x) = \sum_{\alpha \in F} \widehat{f}(\alpha) \cdot \omega^{\alpha x}$.

The following propositions will be useful, which follow directly from the definition.

Proposition 4. *Let $S, T \subseteq F$ be a partition of F. For all $\alpha \in F$,*

$$\widehat{\mathbb{1}_S}(\alpha) = -\widehat{\mathbb{1}_T}(\alpha).$$

Proposition 5 (Properties of Fourier Coefficients). *For all $S \subseteq F$ and $x, \alpha \in F$, it holds that*

$$\widehat{\mathbb{1}_{x+S}}(\alpha) = \widehat{\mathbb{1}_S}(\alpha) \cdot \omega^{-\alpha \cdot x},$$

$$\widehat{\mathbb{1}_S}(x \cdot \alpha) = \widehat{\mathbb{1}_{S \cdot x}}(\alpha).$$

D.2 Some Preparatory Results

The following result rewrites the statistical distance between two leakage distributions using the Fourier coefficients of appropriate indicator functions.

Proposition 6. *Consider* $\mathsf{ShamirSS}(n, n)$ *over a prime field F. Let $C_0^\perp$ be the dual code of* $\mathsf{Share}(0)$. *For any one-bit leakage function, $\vec{\tau}\colon F^n \to \{0,1\}^n$, the following identity holds for any secret $s \in F$.*

$$\begin{aligned} &2\mathsf{SD}\left(\vec{\tau}(\mathsf{Share}(0))\,,\ \vec{\tau}(\mathsf{Share}(s))\right) \\ &\quad = 2^n \left| \sum_{\vec{\gamma} \in \mathcal{C}_0^\perp \setminus \vec{0}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}}_{\tau_i^{-1}(0)}(\gamma_i) \right) \cdot \left(1 - \omega^{s \cdot (\gamma_1 + \cdots + \gamma_n)} \right) \right|. \end{aligned}$$

Proof. The following identity is known in the literature (see [33] for proof).

$$
\begin{aligned}
&2\mathsf{SD}\left(\vec{\tau}(\mathsf{Share}(0))\ ,\ \vec{\tau}(\mathsf{Share}(s))\right) \\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^n} \left| \sum_{\vec{\gamma}\in\mathcal{C}_0^{\perp}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}_{\tau_i^{-1}(\ell_i)}}(\gamma_i) \right) \cdot \left(1-\omega^{s\cdot(\gamma_1+\cdots+\gamma_n)}\right) \right|
\end{aligned}
$$

By Proposition 4, $\widehat{\mathbb{1}_{\tau_i^{-1}(\ell_i)}}(\gamma_i) = \widehat{\mathbb{1}_{\tau_i^{-1}(1-\ell_i)}}(\gamma_i)$ since $\tau_i^{-1}(\ell_i)$ and $\tau_i^{-1}(1-\ell_i)$ are a partition of F. Using this property, one can verify for every $\vec{\ell}, \vec{\ell'} \in \{0,1\}^n$, it holds that

$$
\begin{aligned}
&\left| \sum_{\vec{\gamma}\in\mathcal{C}_0^{\perp}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}_{\tau_i^{-1}(\ell_i)}}(\gamma_i) \right) \cdot \left(1-\omega^{s\cdot(\gamma_1+\cdots+\gamma_n)}\right) \right| \\
&\quad = \left| \sum_{\vec{\gamma}\in\mathcal{C}_0^{\perp}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}_{\tau_i^{-1}(\ell'_i)}}(\gamma_i) \right) \cdot \left(1-\omega^{s\cdot(\gamma_1+\cdots+\gamma_n)}\right) \right|.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&2\mathsf{SD}\left(\vec{\tau}(\mathsf{Share}(0))\ ,\ \vec{\tau}(\mathsf{Share}(s))\right) \\
&\quad = 2^n \left| \sum_{\vec{\gamma}\in\mathcal{C}_0^{\perp}\setminus\vec{0}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}}_{\tau_i^{-1}(0)}(\gamma_i) \right) \cdot \left(1-\omega^{s\cdot(\gamma_1+\cdots+\gamma_n)}\right) \right|,
\end{aligned}
$$

as desired. □

Proposition 7. *Let $A_1, A_2, \ldots, A_n \subseteq F$ and $\beta_1, \beta_2, \ldots, \beta_n \in F^*$. Then, for any $s \in F$, the following identity holds.*

$$
\begin{aligned}
&\sum_{t\in F} \prod_{i=1}^{n} \left(\widehat{\mathbb{1}_{A_i\cdot\beta_i}}(t) \cdot \omega^{s\cdot t\cdot\beta_i} \right) \\
&= \frac{1}{p^{n-1}} \sum_{\substack{x_n\in A_n\cdot\beta_n \\ \vdots \\ x_3\in A_3\cdot\beta_3}} \mathsf{card}(A_2) - \mathsf{card}\left(\left(A_2\cdot\beta_2 + \sum_{i=3}^{n} x_i - s\cdot\sum_{i=1}^{n}\beta_i \right) \bigcap A_1\cdot\beta_1 \right)
\end{aligned}
$$

Proof. We shall extensively use the linear property of Fourier coefficients.

$$
\begin{aligned}
&\sum_{t\in F}\prod_{i=1}^{n}\left(\widehat{\mathbb{1}_{A_i\cdot\beta_i}}(t)\cdot\omega^{s\cdot t\cdot\beta_i}\right)\\
&=\sum_{t\in F}\prod_{i}^{n}\left(\frac{1}{p}\sum_{x_i\in F}\mathbb{1}_{A_i\cdot\beta_i}(x_i)\cdot\omega^{-t\cdot x_i}\cdot\omega^{s\cdot t\cdot\beta_i}\right) && \text{(Fourier expansion)}\\
&=\frac{1}{p^n}\sum_{t\in F}\sum_{\vec{x}\in F^n}\left(\prod_{i=1}^{n}\mathbb{1}_{A_i\cdot\beta_i}(x_i)\cdot\omega^{-t\cdot x_i}\cdot\omega^{s\cdot t\cdot\beta_i}\right) && \text{(Linearity)}\\
&=\frac{1}{p^n}\sum_{\vec{x}\in F^n}\left(\prod_{i=1}^{n}\mathbb{1}_{A_i\cdot\beta_i}(x_i)\right)\sum_{t\in F}\omega^{-t\cdot(x_1+\cdots+x_n-s\cdot(\beta_1+\cdots+\beta_n))} && \text{(Linearity)}\\
&=\frac{1}{p^{n-1}}\sum_{\substack{\vec{x}\in F^n:\\ x_1+\cdots+x_n=s\cdot(\beta_1+\cdots+\beta_n)}}\left(\prod_{i=1}^{n}\mathbb{1}_{A_i\cdot\beta_i}(x_i)\right) && \text{(Sum of roots of unity)}
\end{aligned}
$$

Now, replacing $x_1 = s\cdot(\beta_1+\cdots+\beta_n)-(x_2+\cdots+x_n)$ yields

$$
\begin{aligned}
&\frac{1}{p^{n-1}}\sum_{x_2,\ldots,x_n\in F}\mathbb{1}_{A_1\cdot\beta_1}(s\cdot(\beta_1+\cdots+\beta_n)-(x_2+\cdots+x_n))\cdot\prod_{i=2}^{n}\mathbb{1}_{A_i\cdot\beta_i}(x_i)\\
&=\frac{1}{p^{n-1}}\sum_{\substack{x_n\in A_n\cdot\beta_n\\ \vdots\\ x_3\in A_3\cdot\beta_3}}\sum_{x_2\in F}\mathbb{1}_{A_2\cdot\beta_2}(x_2)\cdot\mathbb{1}_{A_1\cdot\beta_1}(s\cdot(\beta_1+\cdots+\beta_n)-(x_2+\cdots+x_n))
\end{aligned}
$$

Let us take a detour and simplify the inner summand using linear properties of sets and indicator functions as follows.

$$
\begin{aligned}
&\sum_{x_2\in F}\mathbb{1}_{A_2\cdot\beta_2}(x_2)\cdot\mathbb{1}_{A_1\cdot\beta_1}(s\cdot(\beta_1+\cdots+\beta_n)-(x_2+\cdots+x_n))\\
&=\sum_{x_2\in F}\mathbb{1}_{A_2\cdot\beta_2}(x_2)\cdot\mathbb{1}_{A_1\cdot\beta_1-s\cdot(\beta_1+\cdots+\beta_n)+(x_3+\cdots+x_n)}(-x_2)\\
&=\sum_{x_2\in F}\mathbb{1}_{A_2\cdot\beta_2}(x_2)\cdot\mathbb{1}_{-A_1\cdot\beta_1+s\cdot(\beta_1+\cdots+\beta_n)-(x_3+\ldots+x_n)}(x_2)\\
&=\mathsf{card}(A_2\cdot\beta_2\cap(-A_1\cdot\beta_1-(x_3+\ldots+x_n)+s\cdot(\beta_1+\cdots+\beta_n)))\\
&=\mathsf{card}(A_2\cdot\beta_2+(x_3+\cdots+x_n)-s\cdot(\beta_1+\cdots+\beta_n)\cap(-A_1\cdot\beta_1))\\
&=\mathsf{card}(A_2\cdot\beta_2)-\mathsf{card}(A_2\cdot\beta_2+(x_3+\cdots+x_n)-s\cdot(\beta_1+\cdots+\beta_n)\cap A_1\cdot\beta_1)\\
&=\mathsf{card}(A_2)-\mathsf{card}(A_2\cdot\beta_2+(x_3+\cdots+x_n)-s\cdot(\beta_1+\cdots+\beta_n)\cap A_1\cdot\beta_1)\,,
\end{aligned}
$$

which completes the proof. □

D.3 Putting things together and proving Theorem 2

We begin with some notations. Let $\vec{\tau}\colon F^n \to \{0,1\}^n$ be any one-bit physical leakage. Let $A_i = \tau_i^{-1}(0)$ for $1 \leqslant i \leqslant n$. By Imported Theorem 1, the dual code $C_0^{\perp}$ is the set $\{t \cdot (\beta_1, \beta_2, \ldots, \beta_n) \colon t \in F\}$, where

$$\beta_i = \left(\alpha_i \prod_{j\neq i}(\alpha_i - \alpha_j)\right)^{-1}, \text{ for every } i \in \{1,2,\ldots,n\}.$$

Consider the following manipulation.

$$\begin{aligned}
&2\mathsf{SD}\left(\vec{\tau}(\mathsf{Share}(0))\ ,\ \vec{\tau}(\mathsf{Share}(s))\right)\\
&= 2^n \cdot \left|\sum_{\vec{\gamma}\in\mathcal{C}_0^{\perp}\setminus\vec{0}} \prod_{i=1}^{n} \widehat{\mathbb{1}_{\tau_i^{-1}(0)}}(\gamma_i)\cdot\left(1-\omega^{s\cdot(\gamma_1+\cdots+\gamma_n)}\right)\right| \qquad (\textit{Proposition 6})\\
&= 2^n \cdot \left|\sum_{t\in F^*} \prod_{i=1}^{n} \widehat{\mathbb{1}_{A_i}}(t\cdot\beta_i)\cdot\left(1-\omega^{s\cdot t\cdot(\beta_1+\cdots+\beta_n)}\right)\right|\\
&= 2^n \cdot \left|\sum_{t\in F^*} \prod_{i=1}^{n} \widehat{\mathbb{1}_{A_i\cdot\beta_i}}(t) - \sum_{t\in F^*} \prod_{i=1}^{n} \widehat{\mathbb{1}_{A_i\cdot\beta_i}}(t)\cdot\omega^{s\cdot t\cdot\beta_i}\right|
\end{aligned}$$

For each $s \in F$ and tuple $(x_3, x_4, \ldots, x_n)$ satisfying $x_i \in A_i \cdot \beta_i$ for $3 \leqslant i \leqslant n$, we define

$$\varphi_{s,\vec{\tau}}(x_3,x_4,\ldots,x_n) := \sum_{x_n\in A_n\cdot\beta_n}\cdots\sum_{x_3\in A_3\cdot\beta_3} \mathsf{card}\left(\left(A_2\cdot\beta_2+\sum_{i=3}^{n}x_i - s\cdot\sum_{i=1}^{n}\beta_i\right)\bigcap A_1\cdot\beta_1\right).$$

Then, it follows from Proposition 7 that

$$2\mathsf{SD}\left(\vec{\tau}(\mathsf{Share}(0))\ ,\ \vec{\tau}(\mathsf{Share}(s))\right) = \frac{2^{n-1}}{p^{n-1}}\cdot\left|\varphi_{0,\vec{\tau}}(x_3,\ldots,x_n) - \varphi_{s,\vec{\tau}}(x_3,\ldots,x_n)\right|.$$

It suffices to prove the result when $\vec{\tau} = \vec{\mathrm{LSB}}$ (the proof for arbitrary physical bit leakage is similar). In this case, note that $A_1 = A_2 = E = F^+ \cdot 2$. Therefore, we have

$$\begin{aligned}
&\mathsf{card}\left(\left(A_2\cdot\beta_2+\sum_{i=3}^{n}x_i - s\cdot\sum_{i=1}^{n}\beta_i\right)\bigcap A_1\cdot\beta_1\right)\\
&\quad= \mathsf{card}\left(\left(F^+\cdot 2\cdot\beta_2+\sum_{i=3}^{n}x_i - s\cdot\sum_{i=1}^{n}\beta_i\right)\bigcap F^+\cdot 2\cdot\beta_1\right)\\
&\quad= \mathsf{card}\left(\left(F^+\cdot\beta_2+2^{-1}\cdot\left(\sum_{i=3}^{n}x_i - s\cdot\sum_{i=1}^{n}\beta_i\right)\right)\bigcap F^+\cdot\beta_1\right)\\
&\quad= \varSigma_{\beta_1^{-1},\beta_2^{-1}}^{\left(\varDelta^{(s)}_{x_3,\ldots,x_n}\right)},
\end{aligned}$$

where $\Delta^{(s)}_{x_3,\ldots,x_n} := 2^{-1} \cdot (\sum_{i=3}^{n} x_i - s \cdot \sum_{i=1}^{n} \beta_i)$. Similar to the proof of Lemma 2 in Supporting Material C.1, we have

$$\begin{aligned}
&2\mathsf{SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right)\\
&= \frac{2^{n-2}}{p^{n-1}} \cdot \left| \sum_{x_n \in E\cdot\beta_n} \cdots \sum_{x_3 \in E\cdot\beta_3} \left(\Sigma^{\left(\Delta^{(0)}_{x_3,\ldots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} - \Sigma^{\left(\Delta^{(s)}_{x_3,\ldots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} \right) \right| \\
&\leqslant \frac{2^{n-2}}{p^{n-1}} \cdot \sum_{x_n \in E\cdot\beta_n} \cdots \sum_{x_3 \in E\cdot\beta_3} \left| \left(\Sigma^{\left(\Delta^{(0)}_{x_3,\ldots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} - \Sigma^{\left(\Delta^{(s)}_{x_3,\ldots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} \right) \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(By triangle inequality)}
\end{aligned}$$

Suppose $\mathsf{ShamirSS}(2, 2, (\beta_1, \beta_2))$ have ε insecurity against LSB. Then, it follows from Lemma 2 that

$$\left| \Sigma^{\left(\Delta^{(0)}_{x_3,\ldots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} - \Sigma^{\left(\Delta^{(s)}_{x_3,\ldots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} \right| \leqslant 2\varepsilon p. \tag{21}$$

Applying the above equation for every term under the summand yields.

$$\begin{aligned}
2\mathsf{SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right) &\leqslant \frac{2^{n-2}}{p^{n-1}} \cdot \sum_{x_n \in E\cdot\beta_n} \cdots \sum_{x_3 \in E\cdot\beta_3} 2\varepsilon p \\
&\leqslant \frac{2^{n-2}}{p^{n-1}} \cdot \underbrace{(p/2)\cdots(p/2)}_{(n-2)\text{-times}} \cdot 2\varepsilon p \\
&= 2\varepsilon,
\end{aligned}$$

which completes the proof.

E Proof of Lemma 9

Consider the $\mathsf{ShamirSS}(3, 2, (\alpha_1, \alpha_2, \alpha_3))$ secret-sharing scheme over a prime field F_p. Let $s \in F$ be an arbitrary secret. Let us begin by proving the insecurity of $\mathsf{ShamirSS}(3, 2, (\alpha_1, \alpha_2, \alpha_3))$ against LSB leakage attack and then generalize to arbitrary physical bit leakage attack.

E.1 Against LSB Leakage

$$
\begin{aligned}
&2\mathsf{SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right)\\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^3}\left|\Pr\left[\vec{\mathrm{LSB}}(\mathsf{Share}(0)) = \vec{\ell}\right] - \Pr\left[\vec{\mathrm{LSB}}(\mathsf{Share}(s)) = \vec{\ell}\right]\right|\\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^3}\Bigg|\mathop{\mathbb{E}}_X\left[\mathbb{1}_{\mathrm{LSB}^{-1}(\ell_1)}(\alpha_1 X)\cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_2)}(\alpha_2 X)\cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_3)}(\alpha_3 X)\right]\\
&\qquad\qquad - \mathop{\mathbb{E}}_X\left[\mathbb{1}_{\mathrm{LSB}^{-1}(\ell_1)}(\alpha_1 X + s)\cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_2)}(\alpha_2 X + s)\cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_3)}(\alpha_3 X + s)\right]\Bigg|\\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^3}\left|\mathop{\mathbb{E}}_X\left[\prod_{i=1}^{3}\frac{\left(1+(-1)^{\ell_i}\,\mathrm{sign}_p(\alpha_i X\cdot 2^{-1})\right)}{2}\right] - \mathop{\mathbb{E}}_X\left[\prod_{i=1}^{3}\frac{\left(1+(-1)^{\ell_i}\,\mathrm{sign}_p((\alpha_i X + s)\cdot 2^{-1})\right)}{2}\right]\right|\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{Claim 1})\\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^3}\left|\mathop{\mathbb{E}}_Y\left[\prod_{i=1}^{3}\frac{\left(1+(-1)^{\ell_i}\,\mathrm{sign}_p(\alpha_i Y)\right)}{2}\right] - \mathop{\mathbb{E}}_Y\left[\prod_{i=1}^{3}\frac{\left(1+(-1)^{\ell_i}\,\mathrm{sign}_p(\alpha_i Y + t)\right)}{2}\right]\right|\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (X\cdot 2^{-1}\mapsto Y, t = s\cdot 2^{-1})\\
&\quad = \frac{1}{8}\cdot\frac{1}{p}\sum_{\vec{\ell}\in\{0,1\}^3}\left|\sum_{Y\in F_p}\prod_{i=1}^{3}\left(1+(-1)^{\ell_i}\cdot \mathrm{sign}_p(\alpha_i Y)\right) - \sum_{Y\in F}\prod_{i=1}^{3}\left(1+(-1)^{\ell_i}\cdot \mathrm{sign}_p(\alpha_i Y + t)\right)\right|
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\prod_{i=1}^{3}\left(1+(-1)^{\ell_i}\cdot\mathrm{sign}_p(\alpha_i Y + t)\right) = 1 &+ \left(\sum_{i=1}^{3}(-1)^{\ell_i}\cdot\mathrm{sign}_p(\alpha_i Y + t)\right)\\
&+ \left(\sum_{i<j}(-1)^{\ell_i+\ell_j}\cdot\mathrm{sign}_p(\alpha_i Y + t)\,\mathrm{sign}_p(\alpha_j Y + t)\right)\\
&+ (-1)^{\ell_1+\ell_2+\ell_3}\cdot\mathrm{sign}_p(\alpha_1 Y + t)\,\mathrm{sign}_p(\alpha_2 Y + t)\,\mathrm{sign}_p(\alpha_3 Y + t)
\end{aligned}
$$

Since for $\alpha_i, t, Y \in F_p$, $\alpha_i\cdot Y + t$ is an automorphism on F, then for all $\alpha_i, t\in F$

$$\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_i Y + t) = 1.$$

Hence,

$$
\begin{aligned}
&2\mathsf{SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right) \\
&= \frac{1}{8p}\cdot\sum_{\vec{\ell}\in\{0,1\}^3}\Bigg|\left(\sum_{1\leqslant i<j\leqslant 3}(-1)^{\ell_i+\ell_j}\cdot\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_i Y)\,\mathrm{sign}_p(\alpha_j Y)\right) \\
&\qquad -\left(\sum_{1\leqslant i<j\leqslant 3}(-1)^{\ell_i+\ell_j}\cdot\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_i Y+t)\,\mathrm{sign}_p(\alpha_j Y+t)\right) \\
&\qquad +(-1)^{\ell_1+\ell_2+\ell_3}\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_1 Y)\,\mathrm{sign}_p(\alpha_2 Y)\,\mathrm{sign}_p(\alpha_3 Y) \\
&\qquad -(-1)^{\ell_1+\ell_2+\ell_3}\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_1 Y+t)\,\mathrm{sign}_p(\alpha_2 Y+t)\,\mathrm{sign}_p(\alpha_3 Y+t)\Bigg| \\
&=\frac{1}{8p}\cdot\sum_{\vec{\ell}\in\{0,1\}^3}\Bigg|\sum_{1\leqslant i<j\leqslant 3}(-1)^{\ell_i+\ell_j}\cdot\left(\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_i Y)\,\mathrm{sign}_p(\alpha_j Y)-\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_i Y+t)\,\mathrm{sign}_p(\alpha_j Y+t)\right) \\
&\qquad +(-1)^{\ell_1+\ell_2+\ell_3}\cdot\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_1 Y)\,\mathrm{sign}_p(\alpha_2 Y)\,\mathrm{sign}_p(\alpha_3 Y) \\
&\qquad -(-1)^{\ell_1+\ell_2+\ell_3}\cdot\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_1 Y+t)\,\mathrm{sign}_p(\alpha_2 Y+t)\,\mathrm{sign}_p(\alpha_3 Y+t)\Bigg| \\
&=\frac{1}{8p}\cdot\sum_{\vec{\ell}\in\{0,1\}^3}\left|\sum_{1\leqslant i<j\leqslant 3}(-1)^{\ell_i+\ell_j}\cdot\left(\Sigma^{(0)}_{\alpha_i,\alpha_j}-\Sigma^{(\Delta_{i,j})}_{\alpha_i,\alpha_j}\right)+(-1)^{\ell_1+\ell_2+\ell_3}\cdot\left(\Sigma^{(0,0)}_{\alpha_1,\alpha_2,\alpha_3}-\Sigma^{(\Delta_{1,2},\Delta_{1,3})}_{\alpha_1,\alpha_2,\alpha_3}\right)\right| \\
&\leqslant\frac{1}{p}\cdot\sum_{1\leqslant i<j\leqslant 3}\left|\Sigma^{(0)}_{\alpha_i,\alpha_j}-\Sigma^{(\Delta_{i,j})}_{\alpha_i,\alpha_j}\right|+\frac{1}{p}\cdot\left|\Sigma^{(0,0)}_{\alpha_1,\alpha_2,\alpha_3}-\Sigma^{(\Delta_{1,2},\Delta_{1,3})}_{\alpha_1,\alpha_2,\alpha_3}\right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(Triangle Inequiality)}
\end{aligned}
$$

where $\Delta_{i,j} := (s\cdot 2^{-1})\cdot(\alpha_i^{-1}-\alpha_j^{-1})$ for all $1\leqslant i<j\leqslant 3$.

Then,

$$
\begin{aligned}
\mathsf{SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right) &\leqslant \sum_{1\leqslant i<j\leqslant 3}\frac{\left|\Sigma^{(0)}_{\alpha_i,\alpha_j}-\Sigma^{(\Delta_{i,j})}_{\alpha_i,\alpha_j}\right|}{2p}+\frac{\left|\Sigma^{(0,0)}_{\alpha_1,\alpha_2,\alpha_3}-\Sigma^{(\Delta_{1,2},\Delta_{1,3})}_{\alpha_1,\alpha_2,\alpha_3}\right|}{2p} \\
&=\sum_{1\leqslant i<j\leqslant 3}\varepsilon_{\mathrm{LSB}}(\alpha_i,\alpha_j)+\frac{\left|\Sigma^{(0,0)}_{\alpha_1,\alpha_2,\alpha_3}-\Sigma^{(\Delta_{1,2},\Delta_{1,3})}_{\alpha_1,\alpha_2,\alpha_3}\right|}{2p}
\end{aligned}
$$

Consider the following generalization of Lemma 3.

Claim 13 *For $k, \ell, m \in \{1, 2, \dots\}$ and $\Delta, \Delta' \in \{0, 1, \dots, (p-1)\}$,*

$$\Sigma_{k,\ell,m}^{(\Delta,\Delta')} = \widetilde{\Sigma}_{k,\ell,m}^{(\Delta,\Delta')} + \left(\sum_{T \in \{0,\Delta,\Delta'\}} \mathrm{sign}_p(kT) \cdot \mathrm{sign}_p(\ell(T-\Delta)) \cdot \mathrm{sign}_p(m(T-\Delta')) \right).$$

Proof.

$$\begin{aligned}
\Sigma_{k,\ell,m}^{(\Delta,\Delta')} &= \sum_{X \in F_p} \mathrm{sign}_p(k \cdot X) \cdot \mathrm{sign}_p(\ell \cdot (X-\Delta)) \cdot \mathrm{sign}_p(m \cdot (X-\Delta')) \\
&= \sum_{X \in F_p} \widetilde{\mathrm{sign}}_p(k \cdot X) \cdot \widetilde{\mathrm{sign}}_p(\ell \cdot (X-\Delta)) \cdot \widetilde{\mathrm{sign}}_p(m \cdot (X-\Delta')) \\
&\qquad + \sum_{X \in \{0,\Delta,\Delta'\}} \mathrm{sign}_p(k \cdot X) \cdot \mathrm{sign}_p(\ell \cdot (X-\Delta)) \cdot \mathrm{sign}_p(m \cdot (X-\Delta')) \\
&= \widetilde{\Sigma}_{k,\ell,m}^{(\Delta,\Delta')} + \left(\sum_{T \in \{0,\Delta,\Delta'\}} \mathrm{sign}_p(kT) \cdot \mathrm{sign}_p(\ell(T-\Delta)) \cdot \mathrm{sign}_p(m(T-\Delta')) \right)
\end{aligned}$$

□

Claim 14 *For $k, \ell, m \in \{1, 2, \dots\}$ and $\Delta, \Delta' \in \{0, 1, \dots, (p-1)\}$,*

$$\widetilde{\Sigma}_{k,\ell,m}^{(\Delta,\Delta')} = 0.$$

Proof.

$$\begin{aligned}
\widetilde{\Sigma}_{k,\ell,m}^{(\Delta,\Delta')} &= \sum_{X \in F_p} \widetilde{\mathrm{sign}}_p(k \cdot X) \cdot \widetilde{\mathrm{sign}}_p(\ell \cdot (X-\Delta)) \cdot \widetilde{\mathrm{sign}}_p(m \cdot (X-\Delta')) \\
&= \sum_{X \in F_p} \varphi(k \cdot X/p) \cdot \varphi(\ell \cdot (X-\Delta)/p) \cdot \varphi(m \cdot (X-\Delta')/p) \\
&\qquad\qquad (\widetilde{\mathrm{sign}}_p(X) = \varphi(X/p)) \\
&= \sum_{x \in \{\frac{0}{p}, \frac{1}{p}, \dots, \frac{p-1}{p}\}} \varphi(k \cdot x) \cdot \varphi(\ell \cdot (x - \Delta/p)) \cdot \varphi(m \cdot (x - \Delta'/p))
\end{aligned}$$

Recall the Fourier expansion of $\varphi(x)$ is as follows.

$$\varphi(x) = \sum_{\text{odd } n > 0} \frac{4}{\pi n} \cdot \sin(2n\pi x). \tag{22}$$

Substituting $\varphi(x)$ in the expression for $\widetilde{\Sigma}_{k,\ell,m}^{(\Delta,\Delta')}$ with Equation 22,

$$\begin{aligned}
&\widetilde{\Sigma}_{k,\ell,m}^{(\Delta,\Delta')} \\
&= \sum_{x \in \{\frac{0}{p}, \frac{1}{p}, \dots, \frac{p-1}{p}\}} \sum_{\text{odd } n_1, n_2, n_3 > 0} \frac{4^3}{\pi^3 n_1 n_2 n_3} \cdot \sin(2n_1\pi k x) \cdot \sin(2n_2\pi\ell \cdot (x - \Delta/p)) \cdot \sin(2n_3\pi m \cdot (x - \Delta'/p))
\end{aligned}$$

Consider the following trignometric identity,

$$\sin A \cdot \sin B \cdot \sin C = \frac{\sin(A-B+C) - \sin(A-B-C) - \sin(A+B+C) + \sin(A+B-C)}{4}.$$

Substituting $A = 2n_1\pi kx, B = 2n_2\pi\ell \cdot (x - \Delta/p), C = 2n_3\pi m \cdot (x - \Delta'/p)$, we get

$$\begin{aligned}
&4 \cdot \sin(2n_1\pi kx) \cdot \sin(2n_2\pi\ell \cdot (x - \Delta/p)) \cdot \sin(2n_3\pi m \cdot (x - \Delta'/p)) \\
&\quad = \sin(2\pi x \cdot (n_1k - n_2\ell + n_3m) + 2\pi \cdot (n_2\ell\Delta - n_3m\Delta')/p) \\
&\qquad - \sin(2\pi x \cdot (n_1k - n_2\ell - n_3m) + 2\pi \cdot (n_2\ell\Delta + n_3m\Delta')/p) \\
&\qquad - \sin(2\pi x \cdot (n_1k + n_2\ell + n_3m) + 2\pi \cdot (-n_2\ell\Delta - n_3m\Delta')/p) \\
&\qquad + \sin(2\pi x \cdot (n_1k + n_2\ell - n_3m) + 2\pi \cdot (-n_2\ell\Delta + n_3m\Delta')/p)
\end{aligned}$$

Define $a_1 = n_1k - n_2\ell + n_3m, a_2 = n_1k - n_2\ell - n_3m, a_3 = n_1k + n_2\ell + n_3m, a_4 = n_1k + n_2\ell - n_3m$ where $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ and define $b_1 = n_2\ell\Delta - n_3m\Delta', b_2 = n_2\ell\Delta + n_3m\Delta', b_3 = n_2\ell\Delta + n_3m\Delta', b_4 = -n_2\ell\Delta + n_3m\Delta'$ where $b_1, b_2, b_3, b_4 \in \mathbb{Z}$ as well.

$$\begin{aligned}
&\sum_{x \in \{\frac{0}{p}, \frac{1}{p}, \ldots, \frac{p-1}{p}\}} \sin(2\pi \cdot a_1x + 2\pi \cdot b_1/p) \\
=& \sum_{y \in \{\frac{0}{p}, \frac{1}{p}, \ldots, \frac{p-1}{p}\}} \sin(2\pi \cdot (y + b_1/p)) && (a_1 \in \mathbb{Z}) \\
=& \sum_{y \in \{\frac{0}{p}, \frac{1}{p}, \ldots, \frac{p-1}{p}\}} \sin(2\pi \cdot y) && (b_1/p \in 1/p \cdot \mathbb{Z}) \\
=& 0
\end{aligned}$$

Note that the last equality holds because for all $i \in \{1, 2, \ldots, (p-1)/2\}$, we have

$$\sin(2\pi \cdot (p-i)/p) = -\sin(2\pi \cdot i/p).$$

Similarly, we can obtain that

$$\begin{aligned}
\sum_{x \in \{\frac{0}{p}, \frac{1}{p}, \ldots, \frac{p-1}{p}\}} \sin(2\pi \cdot a_2x + 2\pi \cdot b_2/p) &= 0 \\
\sum_{x \in \{\frac{0}{p}, \frac{1}{p}, \ldots, \frac{p-1}{p}\}} \sin(2\pi \cdot a_3x + 2\pi \cdot b_3/p) &= 0 \\
\sum_{x \in \{\frac{0}{p}, \frac{1}{p}, \ldots, \frac{p-1}{p}\}} \sin(2\pi \cdot a_4x + 2\pi \cdot b_4/p) &= 0
\end{aligned}$$

Combining all terms, we get

$$\sum_{x\in\{\frac{0}{p},\frac{1}{p},\dots,\frac{p-1}{p}\}} \sin(2n_1\pi kx)\cdot\sin(2n_2\pi\ell\cdot(x-\Delta/p))\cdot\sin(2n_3\pi m\cdot(x-\Delta'/p))=0$$

which implies that

$$\widetilde{\Sigma}_{k,\ell,m}^{(\Delta,\Delta')}=0.$$

□

Apply Claim 14 to Claim 13, we get

Claim 15 *For $k,\ell,m\in\{1,2,\dots\}$ and $\Delta,\Delta'\in\{0,1,\dots,(p-1)\}$,*

$$\Sigma_{k,\ell,m}^{(\Delta,\Delta')}=\left(\sum_{T\in\{0,\Delta,\Delta'\}}\mathrm{sign}_p(kT)\cdot\mathrm{sign}_p(\ell(T-\Delta))\cdot\mathrm{sign}_p(m(T-\Delta'))\right).$$

Claim 15 implies that

$$\Sigma_{\alpha_1,\alpha_2,\alpha_3}^{(0,0)}-\Sigma_{\alpha_1,\alpha_2,\alpha_3}^{(\Delta_{1,2},\Delta_{1,3})}=-\sum_{T\in\{\Delta_{1,2},\Delta_{1,3}\}}\mathrm{sign}_p(k\cdot T)\cdot\mathrm{sign}_p(\ell\cdot(T-\Delta_{1,2}))\cdot\mathrm{sign}_p(m\cdot(T-\Delta_{1,3})).$$

Then,

$$\left|\Sigma_{\alpha_1,\alpha_2,\alpha_3}^{(0,0)}-\Sigma_{\alpha_1,\alpha_2,\alpha_3}^{(\Delta_{1,2},\Delta_{1,3})}\right|\leqslant 2.$$

Therefore,

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}):=\mathsf{SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right)\leqslant\sum_{1\leqslant i<j\leqslant 3}\varepsilon_{\mathrm{LSB}}(\alpha_i,\alpha_j)+\frac{1}{p}.$$

E.2 Against arbitrary physical bit leakage attack

Let $\mathrm{LSB}_i\colon F\to\{0,1\}$ be defined as in Section 5. $\mathrm{LSB}_i\colon F\to\{0,1\}$ is the function that outputs the i-th least significant bit in the binary representation.

We begin by considering a generalization of Proposition 2.

Claim 16 *For all $i\in\{0,1,\dots,\lambda-1\}$, we have $\mathrm{LSB}_i(x)=\mathrm{LSB}(x\cdot 2^{-i})$ for $x\in F$.*

$$
\begin{aligned}
&2\mathsf{SD}\left(\vec{\mathrm{LSB}}_{i_1,i_2,i_3}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}_{i_1,i_2,i_3}(\mathsf{Share}(s))\right) \\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^3} \left|\Pr\left[\vec{\mathrm{LSB}}_{i_1,i_2,i_3}(\mathsf{Share}(0)) = \vec{\ell}\right] - \Pr\left[\vec{\mathrm{LSB}}_{i_1,i_2,i_3}(\mathsf{Share}(s)) = \vec{\ell}\right]\right| \\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^3} \left|\mathop{\mathbb{E}}_{X}\left[\mathbb{1}_{\mathrm{LSB}_{i_1}^{-1}(\ell_1)}(\alpha_1 X)\cdot \mathbb{1}_{\mathrm{LSB}_{i_2}^{-1}(\ell_2)}(\alpha_2 X)\cdot \mathbb{1}_{\mathrm{LSB}_{i_3}^{-1}(\ell_3)}(\alpha_3 X)\right]\right. \\
&\qquad\qquad \left. - \mathop{\mathbb{E}}_{x}\left[\mathbb{1}_{\mathrm{LSB}_{i_1}^{-1}(\ell_1)}(\alpha_1 X+s)\cdot \mathbb{1}_{\mathrm{LSB}_{i_2}^{-1}(\ell_2)}(\alpha_2 X+s)\cdot \mathbb{1}_{\mathrm{LSB}_{i_3}^{-1}(\ell_3)}(\alpha_3 X+s)\right]\right| \\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^3} \left|\mathop{\mathbb{E}}_{X}\left[\mathbb{1}_{\mathrm{LSB}^{-1}(\ell_1)}(\alpha_1 X\cdot 2^{-i_1})\cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_2)}(\alpha_2 X\cdot 2^{-i_2})\cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_3)}(\alpha_3 X\cdot 2^{-i_3})\right]\right. \\
&\quad \left. - \mathop{\mathbb{E}}_{X}\left[\mathbb{1}_{\mathrm{LSB}^{-1}(\ell_1)}((\alpha_1 X+s)\cdot 2^{-i_1})\cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_2)}((\alpha_2 X+s)\cdot 2^{-i_2})\cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_3)}((\alpha_3 X+s)\cdot 2^{-i_3})\right]\right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(By Claim 16)} \\
&\quad = \sum_{\vec{\ell}\in\{0,1\}^3} \left|\mathop{\mathbb{E}}_{X}\left[\prod_{j=1}^{3}\frac{\left(1+(-1)^{\ell_j}\,\mathrm{sign}_p(\alpha_j X\cdot 2^{-i_j})\right)}{2}\right] - \mathop{\mathbb{E}}_{X}\left[\prod_{j=1}^{3}\frac{\left(1+(-1)^{\ell_j}\,\mathrm{sign}_p((\alpha_j X+s)\cdot 2^{-i_j})\right)}{2}\right]\right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Claim 1)} \\
&\quad = \frac{1}{8}\cdot\frac{1}{p}\sum_{\vec{\ell}\in\{0,1\}^3}\left|\sum_{X\in F_p}\prod_{j=1}^{3}\left(1+(-1)^{\ell_j}\,\mathrm{sign}_p(\alpha_j X\cdot 2^{-i_j})\right)\right. \\
&\qquad\qquad\qquad \left. - \sum_{X\in F_p}\prod_{j=1}^{3}\left(1+(-1)^{\ell_j}\,\mathrm{sign}_p((\alpha_j X+s)\cdot 2^{-i_j})\right)\right|
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\prod_{j=1}^{3}\left(1+(-1)^{\ell_j}\cdot\mathrm{sign}_p((\alpha_j X+s)\cdot 2^{-i_j})\right) \\
&\quad = 1 + \left(\sum_{j=1}^{3}(-1)^{\ell_j}\cdot\mathrm{sign}_p((\alpha_j X+s)\cdot 2^{-i_j})\right) \\
&\qquad + \left(\sum_{1\leqslant j_1<j_2\leqslant 3}(-1)^{\ell_{j_1}+\ell_{j_2}}\cdot\mathrm{sign}_p((\alpha_{j_1}X+s)\cdot 2^{-i_{j_1}})\,\mathrm{sign}_p((\alpha_{j_2}X+s)\cdot 2^{-i_{j_2}})\right) \\
&\qquad + (-1)^{\ell_1+\ell_2+\ell_3}\cdot\mathrm{sign}_p((\alpha_1 X+s)\cdot 2^{-i_1})\,\mathrm{sign}_p((\alpha_2 X+s)\cdot 2^{-i_2})\,\mathrm{sign}_p((\alpha_3 X+s)\cdot 2^{-i_3})
\end{aligned}
$$

Since for $\alpha_j, s, X \in F_p$, $\alpha_j\cdot 2^{-i_j}\cdot X + s\cdot 2^{-i_j}$ is an automorphism on F, then

$$
\sum_{X\in F_p}\mathrm{sign}_p(\alpha_j\cdot 2^{-i_j}\cdot X + s\cdot 2^{-i_j}) = 1.
$$

Hence,

$$
\begin{aligned}
&\mathsf{2SD}\left(\mathrm{L\vec{S}B}(\mathsf{Share}(0))\ ,\ \mathrm{L\vec{S}B}(\mathsf{Share}(s))\right)\\
=&\frac{1}{8p}\cdot\sum_{\vec{\ell}\in\{0,1\}^3}\Bigg|\sum_{1\leqslant j_1<j_2\leqslant 3}(-1)^{\ell_{j_1}+\ell_{j_2}}\cdot\Bigg(\sum_{X\in F_p}\mathrm{sign}_p(\alpha_{j_1}X\cdot 2^{-i_{j_1}})\,\mathrm{sign}_p(\alpha_{j_2}X\cdot 2^{-i_{j_2}})\\
&\qquad-\sum_{X\in F_p}\mathrm{sign}_p((\alpha_{j_1}X+s)\cdot 2^{-i_{j_1}})\,\mathrm{sign}_p((\alpha_{j_2}X+s)\cdot 2^{-i_{j_2}})\Bigg)\\
&+(-1)^{\ell_1+\ell_2+\ell_3}\cdot\Bigg(\sum_{X\in F_p}\mathrm{sign}_p(\alpha_1X\cdot 2^{-i_1})\,\mathrm{sign}_p(\alpha_2X\cdot 2^{-i_2})\,\mathrm{sign}_p(\alpha_3X\cdot 2^{-i_2})\\
&\qquad-\sum_{X\in F_p}\mathrm{sign}_p((\alpha_1X+s)\cdot 2^{-i_1})\,\mathrm{sign}_p((\alpha_2X+s)\cdot 2^{-i_2})\,\mathrm{sign}_p((\alpha_3X+s)\cdot 2^{-i_3})\Bigg)\Bigg|
\end{aligned}
\tag{23}
$$

At this point, we introduce the following variable renaming.

Claim 17

$$
\begin{aligned}
&\mathrm{sign}_p(\alpha_1X\cdot 2^{-i_1}+2^{-i_1}\cdot s)\cdot\mathrm{sign}_p(\alpha_2X\cdot 2^{-i_2}+2^{-i_2}\cdot s)\\
&=\mathrm{sign}_p(\alpha_1Y\cdot 2^{-i_1}+s')\cdot\mathrm{sign}_p(\alpha_2Y\cdot 2^{-i_2}+s')
\end{aligned}
$$

where

$$
Y:=X+\frac{2^{-i_1}-2^{-i_2}}{2^{-i_1}\alpha_1-2^{-i_2}\alpha_2},\qquad \textit{and}\qquad s':=\frac{2^{-i_1}2^{-i_2}(\alpha_1-\alpha_2)}{2^{-i_1}\alpha_1-2^{-i_2}\alpha_2}\cdot s
$$

The proof of this claim is by direct substitution. Note that $s\mapsto s'$ is an automorphism over F^* and s' depends on i_{j_1} and i_{j_2}. Then, for

$$
Y:=X+\frac{2^{-i_{j_1}}-2^{-i_{j_2}}}{2^{-i_{j_1}}\alpha_{j_1}-2^{-i_{j_2}}\alpha_{j_2}},\qquad \text{and}\qquad s':=\frac{2^{-i_{j_1}}2^{-i_{j_2}}(\alpha_{j_1}-\alpha_{j_2})}{2^{-i_{j_1}}\alpha_{j_1}-2^{-i_{j_2}}\alpha_{j_2}}\cdot s
$$

$$
\begin{aligned}
&\sum_{X\in F_p}\mathrm{sign}_p(\alpha_{j_1}X\cdot 2^{-i_{j_1}})\,\mathrm{sign}_p(\alpha_{j_2}X\cdot 2^{-i_{j_2}})-\sum_{X\in F_p}\mathrm{sign}_p((\alpha_{j_1}X+s)\cdot 2^{-i_{j_1}})\,\mathrm{sign}_p((\alpha_{j_2}X+s)\cdot 2^{-i_{j_2}})\\
=&\sum_{X\in F_p}\mathrm{sign}_p(\alpha_{j_1}X\cdot 2^{-i_{j_1}})\,\mathrm{sign}_p(\alpha_{j_2}X\cdot 2^{-i_{j_2}})-\sum_{Y\in F_p}\mathrm{sign}_p(\alpha_{j_1}Y\cdot 2^{-i_{j_1}}+s')\,\mathrm{sign}_p(\alpha_{j_2}Y\cdot 2^{-i_{j_2}}+s')\\
&&\text{(By Claim 17)}\\
=&\sum_{X\in F_p}\mathrm{sign}_p(\alpha'_{j_1}X)\,\mathrm{sign}_p(\alpha'_{j_2}X)-\sum_{Y\in F_p}\mathrm{sign}_p(\alpha'_{j_1}Y+s')\,\mathrm{sign}_p(\alpha'_{j_2}Y+s')\\
&&(\alpha_{j_1}\cdot 2^{-i_{j_1}}\mapsto\alpha'_{j_1}\text{ and }\alpha_{j_2}\cdot 2^{-i_{j_2}}\mapsto\alpha'_{j_2})\\
=&\Sigma^{(0)}_{\alpha'_{j_1},\alpha'_{j_2}}-\Sigma^{(\Delta_{j_1,j_2})}_{\alpha'_{j_1},\alpha'_{j_2}}
\end{aligned}
\tag{24}
$$

where $\varDelta_{j_1,j_2} = (s' \cdot 2^{-1}) \cdot ((\alpha'_{j_1})^{-1} - (\alpha'_{j_2})^{-1})$.

Define $\alpha'_1 := \alpha_1 \cdot 2^{-i_1}, \alpha'_2 := \alpha_2 \cdot 2^{-i_2}, \alpha'_3 := \alpha_3 \cdot 2^{-i_3}$. Then,

$$\begin{aligned}
&\sum_{X \in F_p} \operatorname{sign}_p(\alpha_1 X \cdot 2^{-i_1}) \operatorname{sign}_p(\alpha_2 X \cdot 2^{-i_2}) \operatorname{sign}_p(\alpha_3 X \cdot 2^{-i_2}) \\
&\qquad - \sum_{X \in F_p} \operatorname{sign}_p((\alpha_1 X + s) \cdot 2^{-i_1}) \operatorname{sign}_p((\alpha_2 X + s) \cdot 2^{-i_2}) \operatorname{sign}_p((\alpha_3 X + s) \cdot 2^{-i_3}) \\
=& \sum_{X \in F_p} \operatorname{sign}_p(\alpha'_1 X) \operatorname{sign}_p(\alpha'_2 X) \operatorname{sign}_p(\alpha'_3 X) \\
&\qquad - \sum_{X \in F_p} \operatorname{sign}_p(\alpha'_1 X + s \cdot 2^{-i_1}) \operatorname{sign}_p(\alpha'_2 X + s \cdot 2^{-i_2}) \operatorname{sign}_p(\alpha'_3 X + s \cdot 2^{-i_3}) \\
=& \sum_{X \in F_p} \operatorname{sign}_p(\alpha'_1 X) \operatorname{sign}_p(\alpha'_2 X) \operatorname{sign}_p(\alpha'_3 X) \\
&\qquad - \sum_{Y \in F_p} \operatorname{sign}_p(\alpha'_1 Y) \operatorname{sign}_p(\alpha'_2 (Y - \varDelta)) \operatorname{sign}_p(\alpha'_3 (Y - \varDelta')) \\
&\qquad\qquad (X + s \cdot 2^{-i_1} \cdot (\alpha'_1)^{-1} \mapsto Y) \\
=& \varSigma^{(0,0)}_{\alpha'_1,\alpha'_2,\alpha'_3} - \varSigma^{(\varDelta,\varDelta')}_{\alpha'_1,\alpha'_2,\alpha'_3} \qquad (25)
\end{aligned}$$

where $\varDelta := s \cdot 2^{-i_1} \cdot (\alpha'_1)^{-1} - s \cdot 2^{-i_2} \cdot (\alpha'_2)^{-1}$ and $\varDelta' := s \cdot 2^{-i_1} \cdot (\alpha'_1)^{-1} - s \cdot 2^{-i_3} \cdot (\alpha'_3)^{-1}$.

By Claim 15, we get

$$\begin{aligned}
&\varSigma^{(0,0)}_{\alpha'_1,\alpha'_2,\alpha'_3} - \varSigma^{(\varDelta,\varDelta')}_{\alpha'_1,\alpha'_2,\alpha'_3} \\
&\quad = - \sum_{T \in \{\varDelta, \varDelta'\}} \operatorname{sign}_p(\alpha'_1 \cdot T) \cdot \operatorname{sign}_p(\alpha'_2 \cdot (T - \varDelta)) \cdot \operatorname{sign}_p(\alpha'_3 \cdot (T - \varDelta'))
\end{aligned}$$

Then,

$$\left| \varSigma^{(0,0)}_{\alpha'_1,\alpha'_2,\alpha'_3} - \varSigma^{(\varDelta,\varDelta')}_{\alpha'_1,\alpha'_2,\alpha'_3} \right| \leqslant 2.$$

Substituting Equation 24 and Equation 25 to the expression in Equation 23 as follows.

$$\begin{aligned}
&\mathsf{2SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right) \\
&\quad = \frac{1}{8p} \cdot \sum_{\vec{\ell} \in \{0,1\}^3} \left| \sum_{1 \leqslant j_1 < j_2 \leqslant 3} (-1)^{\ell_{j_1} + \ell_{j_2}} \cdot \left(\varSigma^{(0)}_{\alpha'_{j_1},\alpha'_{j_2}} - \varSigma^{(\varDelta_{j_1,j_2})}_{\alpha'_{j_1},\alpha'_{j_2}} \right) \right. \\
&\qquad\qquad \left. + (-1)^{\ell_1 + \ell_2 + \ell_3} \cdot \left(\varSigma^{(0,0)}_{\alpha'_1,\alpha'_2,\alpha'_3} - \varSigma^{(\varDelta,\varDelta')}_{\alpha'_1,\alpha'_2,\alpha'_3} \right) \right|
\end{aligned}$$

where $\varDelta_{j_1,j_2} = (s' \cdot 2^{-1}) \cdot ((\alpha'_{j_1})^{-1} - (\alpha'_{j_2})^{-1})$, $\varDelta := s \cdot 2^{-i_1} \cdot (\alpha'_1)^{-1} - s \cdot 2^{-i_2} \cdot (\alpha'_2)^{-1}$ and $\varDelta' := s \cdot 2^{-i_1} \cdot (\alpha'_1)^{-1} - s \cdot 2^{-i_3} \cdot (\alpha'_3)^{-1}$.

By triangle inequality,

$$\mathsf{SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right) \leqslant \sum_{1\leqslant j_1<j_2\leqslant 3} \frac{\left|\Sigma^{(0)}_{\alpha'_{j_1},\alpha'_{j_2}} - \Sigma^{(\Delta_{j_1,j_2})}_{\alpha'_{j_1},\alpha'_{j_2}}\right|}{2p} + \frac{1}{p}$$

Define $\varepsilon := \max_{1\leqslant i<j\leqslant 3}\{\varepsilon_{\mathrm{PHYS}}(\alpha_i,\alpha_j)\}$. Thus,

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant 3\cdot\varepsilon + \frac{1}{p}.$$

F Some Technical Results

F.1 Proof of Lemma 8

Define

$$\gamma_1 := \alpha_1 \prod_{j\neq 1}(\alpha_1-\alpha_j)$$
$$\gamma_2 := \alpha_2 \prod_{j\neq 2}(\alpha_2-\alpha_j).$$

First, we will show that $[\gamma_1 : \gamma_2]$ is a random equivalence class when α_3 is chosen randomly (and everything else is arbitrarily fixed).

Toward this objective, fix arbitrary $\alpha_1,\alpha_2 \in F_p^*$ such that $\alpha_1 \neq \alpha_2$, and arbitrary $\alpha_4,\alpha_5,\dots,\alpha_n \in F_p$, such that $\{\alpha_1,\alpha_2\}\cap\{\alpha_4,\dots,\alpha_n\} = \emptyset$. *Consider* $\alpha_3 \leftarrow F_p \setminus \{\alpha_1\}$.

$$[\gamma_1 : \gamma_2] = \left[\alpha_1 \prod_{j\neq 1}(\alpha_1-\alpha_j)\ :\ \alpha_2\prod_{j\neq 2}(\alpha_2-\alpha_j)\right] \quad \text{(by definition)}$$

$$= \left[1:\ -\frac{\alpha_2}{\alpha_1}\cdot\prod_{j\geqslant 3}\left(\frac{\alpha_2-\alpha_j}{\alpha_1-\alpha_j}\right)\right]$$

(because $\alpha_1 \neq 0$ and $\alpha_1 \notin \{\alpha_3,\alpha_4,\dots,\alpha_n\}$)

$$= \left[1\colon \Delta\cdot\left(\frac{\alpha_2-\alpha_3}{\alpha_1-\alpha_3}\right)\right],\ \text{where } \Delta := -\frac{\alpha_2}{\alpha_1}\cdot\prod_{j\geqslant 4}\left(\frac{\alpha_2-\alpha_j}{\alpha_1-\alpha_j}\right)$$

$$= \left[1\colon \underbrace{\Delta\cdot\left(1+\frac{\alpha_2-\alpha_1}{\alpha_1-\alpha_3}\right)}_{\Gamma}\right]$$

We make the following observations.

1. $\Delta \neq 0$, because $\alpha_2 \neq 0$ and $\alpha_2 \notin \{\alpha_4,\dots,\alpha_n\}$.

2. $(\alpha_1 - \alpha_3)$ is a uniform distribution over F_p^*, because $\alpha_1 \neq \alpha_3$.
3. $\frac{\alpha_2-\alpha_1}{\alpha_1-\alpha_3}$ is a uniform distribution over F_p^*, because $\alpha_1 \neq \alpha_2$.
4. $\left(1 + \frac{\alpha_2-\alpha_1}{\alpha_1-\alpha_3}\right)$ is a uniform distribution over $F_p \setminus \{1\}$.
5. Γ is a uniform distribution over $F_p \setminus \{\Delta\}$.

Let Γ' be the uniform distribution over $F_p \setminus \{0, 1\}$. Note that

$$\mathsf{SD}\left(\Gamma\ ,\ \Gamma'\right) \leqslant \frac{2}{p-1}.$$

Therefore, $[1 : \Gamma]$ is $2/(p-1)$-close to a uniform distribution over the equivalence classes $[1 : 2], [1 : 3], \ldots, [1 : p-1]$. Note that $[\beta_1 : \beta_2]$ is identical to $[\gamma_1^{-1} : \gamma_2^{-1}] = \left[1 : \Gamma^{-1}\right]$, which is $2/(p-1)$-close to a uniform distribution over the equivalence classes $[1 : 2], [1 : 3], \ldots, [1 : p-1]$.

G Example of Secure Evaluation places against Physical Bit Leakage

We consider $\mathsf{ShamirSS}(n = 2, k = 2, (\alpha_1, \alpha_2))$ over the prime field F of order $p = 2^\lambda - 1$ – a Mersenne prime. We deduced earlier that the security of (α_1, α_2) is identical to the security of all (u, v) in the equivalence class $[\alpha_1 : \alpha_2]$. Note that $[\alpha_1 : \alpha_2]$ is identical to the equivalence class $[1 : \alpha]$, where $\alpha = \alpha_2\alpha_1^{-1}$. The equivalence class $[1 : \alpha]$ is secure if and only if all the following equivalence classes

$$\left\{[1 : \alpha], [1 : 2^1 \cdot \alpha], [1 : 2^2 \cdot \alpha], \ldots, [1 : 2^{\lambda-1} \cdot \alpha]\right\}$$

are secure against the LSB leakage.

The elements generated by 2, $\langle 2 \rangle = \{1, 2, 2^2, \ldots, 2^{\lambda-1}\}$, is a cyclic subgroup of F^*. Let $\alpha \cdot \langle 2 \rangle$ denote the coset $\{\alpha, 2 \cdot \alpha, \ldots, 2^{\lambda-1} \cdot \alpha\} \in F^*/\langle 2 \rangle$. Furthermore, the equivalence class $[1 : \alpha]$ is secure against arbitrary physical bit leakage if (and only if) the equivalence classes $[1 : \alpha']$ are secure against arbitrary physical bit leakage, for all $\alpha' \in \alpha \cdot \langle 2 \rangle$.

> So, in the table below, when we mention α, it implies that any $(\alpha_1, \alpha_2) \in [1 : \alpha']$ is secure against physical bit leakage attacks, where $\alpha' \in \alpha\langle 2 \rangle$.

Remark 8 (Adversarial LLL: A worst-case analysis). For one (α_1, α_2), there may be multiple $(u, v) \in [\alpha_1, \alpha_2]$ that the LLL algorithm can output. The output of the LLL algorithm is crucial in assessing whether evaluation places are secure. The LLL output can change our algorithm's output in Figure 1 from "secure" to "may be insecure."

For example, consider the prime $p = 127$ and $(\alpha_1, \alpha_2) = (1, 23)$. In this case, $B = \left\lceil 2^{3/4}\sqrt{p} \right\rceil = 19$. Note that $(-11, 1) \in [\alpha_1 : \alpha_2]$ and $(6, 11) \in [\alpha_1 : \alpha_2]$. If the LLL algorithm returns $(11, -1)$, our algorithm will declare "may be insecure." If the LLL algorithm returns $(6, 11)$, our algorithm will declare "secure."

Consider an "adversarial LLL" algorithm implementation for the worst-case evaluation. On input (α_1, α_2), if there is $(u, v) \in [\alpha_1 : \alpha_2]$ that makes our algorithm in Figure 1 output "may be insecure," the adversarial LLL outputs that (u, v).

Consider an example of secure evaluation places for Mersenne prime $p = 2^{13} - 1 = 8191$. The example evaluation places are secure even if the "adversarial LLL" algorithm is used. Our code (running on Intel Core i7 7700K) returns all the secure evaluation places in 45.515 seconds.

For example, the element "95" in Table 1 represents the following. Any $(\alpha_1, \alpha_2) \in [1 : \alpha']$ is secure against physical bit leakage attacks, where $\alpha' \in 95\langle 2\rangle$. Note that

$$\begin{aligned}95 \cdot \langle 2\rangle =& \{95, 2 \cdot 95, 2^2 \cdot 95, \ldots, 2^{12} \cdot 95\} \\ =& \{95, 190, 380, 760, 1520, 3040, 6080, 3969, 7938, 7685, 7179, 6167, 4143\}\end{aligned}$$

Corollary 7 presents explicit evaluation places $(\alpha_1, \alpha_2) \in \left[1\colon 2^{\lfloor \lambda/2 \rfloor} - 1\right]$ such that for security parameter λ,

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{4 \cdot \left(2^{\lfloor \lambda/2 \rfloor} + 2^{\lceil \lambda/2 \rceil}\right) - 6}{p}.$$

When $\lambda = 13$ and $p = 2^{13} - 1$, it implies that $[1 : 63]$ would have $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \lesssim 0.093$. However, $63 \cdot \langle 2\rangle$ is not listed in Table 1 because the "adversarial LLL" algorithm may pick $(u, v) = (1, 63)$ which is characterized as "may be insecure" by our algorithm in Figure 1.

To generalize to $\mathsf{ShamirSS}(3, 2, (\alpha_1, \alpha_2, \alpha_3))$ over the prime field F of order $p = 2^\lambda - 1$, we consider the equivalence class $[1 : \alpha : \alpha']$ where $\alpha = \alpha_2\alpha_1^{-1}$ and $\alpha' = \alpha_3\alpha_1^{-1}$. If α, α' and $\alpha'\alpha^{-1}$ all belong to different cosets in Table 1, then the equivalence class $[1 : \alpha : \alpha']$ is secure against arbitrary physical bit leakage.

For example, $[1 : 95 : 103]$ is a good equivalence class of evaluation places against physical bit leakage attack for $\mathsf{ShamirSS}(3, 2, (\alpha_1, \alpha_2, \alpha_3))$. Consider $\alpha = 95 \in 95 \cdot \langle 2\rangle$ and $\alpha' = 103 \in 103 \cdot \langle 2\rangle$ which are good evaluation places in Table 1. Then, $\alpha'\alpha^{-1} = 6209 \in 225 \cdot \langle 2\rangle$ is also a good evaluation place against physical bit leakage attacks.

Table 2 presents choices of α such that evaluation places in equivalence classes of the form $[1 : 95 : \alpha]$ are secure for $\mathsf{ShamirSS}(3, 2, \vec{\alpha})$. If we choose $\alpha \in \alpha' \cdot \langle 2\rangle$ from one of the cosets in Table 1, we only need to check $\alpha \cdot 95^{-1}$ is also contained in one of the coset.

95	97	99	101	103	107	111	113	119	121	123
125	131	133	135	137	139	143	145	147	151	153
155	157	159	161	163	165	169	173	175	179	181
183	185	187	191	197	201	203	207	209	211	213
215	217	219	221	223	225	227	229	231	233	235
237	239	243	245	247	249	251	253	267	269	271
275	277	279	281	285	287	291	293	295	297	299
303	305	309	313	317	319	323	325	329	331	333
335	337	339	349	351	355	357	359	361	363	365
369	371	373	375	377	379	391	393	395	397	399
401	403	405	407	411	413	415	419	423	427	429
433	435	437	441	443	445	447	453	457	459	461
465	467	469	471	473	475	477	487	491	493	495
497	499	501	503	505	549	551	553	555	557	559
563	567	569	573	575	581	583	587	589	591	595
599	601	603	607	611	613	615	617	619	621	623
629	633	637	651	653	655	661	667	669	671	675
677	679	687	693	695	697	699	701	713	715	717
719	725	727	729	731	735	739	743	747	751	755
757	759	761	763	795	797	799	805	807	811	813
815	821	823	825	829	843	845	847	855	857	859
863	869	871	873	875	877	879	883	885	887	889
891	893	915	917	921	923	925	927	933	937	939
943	947	949	951	953	955	957	959	971	973	975
979	987	989	991	997	1001	1005	1007	1011	1175	1181
1183	1191	1197	1199	1205	1207	1211	1213	1227	1231	1235
1237	1239	1245	1247	1253	1255	1259	1261	1263	1267	1275
1323	1327	1333	1335	1339	1341	1343	1355	1357	1359	1371
1373	1375	1387	1389	1395	1397	1403	1405	1431	1435	1439
1447	1451	1461	1467	1469	1485	1487	1491	1495	1499	1501
1503	1511	1515	1519	1525	1655	1661	1691	1693	1695	1703
1709	1711	1717	1723	1725	1727	1743	1751	1757	1759	1773
1775	1783	1787	1851	1853	1855	1871	1879	1885	1887	1899
1901	1903	1909	1915	1963	1965	1967	1973	1975	1979	1981
1983	2007	2011	2013	2015	2775	2783	2795	2799	2807	2911
2927	2935	2939	2991	2999	3003	3035	3039	3055	3551	3575

Table 1. Secure Evaluation Places against Physical Bit Leakage when $p = 2^{13} - 1$. If an element $\alpha \in F$ appears in the list above, it implies the following. Any evaluation places $(\alpha_1, \alpha_2) \in [1 : \alpha']$, where $\alpha' \in \alpha \cdot \langle 2 \rangle$, is secure against all physical bit leakage attacks.

97	99	103	111	113	119	121	125	135	139	143
151	155	159	165	173	175	181	185	187	191	203
207	215	217	225	229	231	233	235	237	239	243
245	251	269	271	275	277	279	281	291	293	295
297	299	305	309	313	317	325	331	335	339	349
351	355	357	361	363	365	371	373	377	379	391
393	395	397	399	403	405	407	413	415	429	435
437	445	447	457	459	461	467	469	471	473	477
487	491	495	497	499	501	503	505	551	553	555
559	575	581	583	603	607	611	613	615	617	621
623	637	651	653	655	661	667	671	679	687	693
695	697	701	713	715	719	725	729	735	743	755
757	797	799	805	807	811	813	815	823	825	829
843	847	857	859	863	869	871	873	877	879	883
885	891	893	915	921	923	937	939	947	951	955
959	973	975	987	989	991	997	1005	1007	1011	1175
1197	1199	1205	1207	1211	1213	1227	1231	1237	1239	1245
1247	1253	1259	1261	1275	1327	1335	1341	1355	1357	1371
1373	1389	1397	1403	1405	1447	1451	1461	1467	1469	1485
1495	1511	1519	1525	1691	1693	1695	1703	1709	1711	1723
1725	1743	1751	1757	1783	1851	1853	1855	1871	1885	1903
1909	1915	1963	1965	1973	1975	1979	1983	2013	2795	2807
2911	2935	2939	2991	2999	3035					

Table 2. Secure Evaluation Places against Physical Bit Leakage when $p = 2^{13} - 1$ and $(n, k) = (3, 2)$. If an element $\alpha \in F$ appears in the list above, it implies the following. Any evaluation places $(\alpha_1, \alpha_2, \alpha_3) \in [1 : 95 : \alpha']$, where $\alpha' \in \alpha \cdot \langle 2 \rangle$, is secure against all physical bit leakage attacks.